On the approximation property for Banach spaces predual to H^∞-spaces

Alexander Brudnyi
Department of Mathematics and Statistics
University of Calgary, Calgary
Canada

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Definition 1

A Banach space X is said to have the *approximation property*, if, for every compact set $K \subset X$ and every $\varepsilon > 0$, there exists an operator $T : X \to X$ of finite rank so that $\|Tx - x\| \le \varepsilon$ for every $x \in K$.

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Although it is strongly believed that the class of spaces with the approximation property includes practically all spaces which appear naturally in analysis, it is not known yet even for the space $H^\infty := H^\infty(\mathbb{D})$. The strongest result in this direction due to Bourgain and Reinov states that H^∞ has the approximation property "up to logarithm". The first example of a space which fails to have the approximation property was constructed by Enflo. Since Enflo's work several other examples of such spaces were constructed.

Definition 2

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Every Banach spaces with a basis has the bounded approximation property. Figiel and Johnson proved that the approximation property does not imply the bounded approximation property. Also, it was established by Pełczyński that a separable Banach space has the bounded approximation property if and only if it is isomorphic to a complemented subspace of a separable Banach space with a basis.

Next, for Banach spaces X, Y by $\mathcal{L}(X, Y)$ and $\mathcal{F}(X, Y)$ we denote the spaces of linear bounded operators and operators of finite rank $X \to Y$ equipped with the operator norm. Let us consider the trace mapping V from the projective tensor product $Y^* \hat{\otimes}_\pi X \to \mathcal{F}(X, Y)^*$ defined by

$$(Vu)(T) = \operatorname{trace}(Tu), \quad \text{where} \quad u \in Y^* \hat{\otimes}_\pi X,\ T \in \mathcal{F}(X, Y),$$

that is, if $u = \sum_{n=1}^{\infty} y_n^* \otimes x_n$, then $(Vu)(T) = \sum_{n=1}^{\infty} y_n^*(Tx_n)$.
It is easy to see that $\|Vu\| \le \|u\|_\pi$. The λ-bounded approximation property of X is equivalent to the fact that $\|u\|_\pi \le \lambda \|Vu\|$ for all Banach spaces Y. This well-known result is essentially due to Grothendieck.

Let M be a (finite-dimensional) Caratheodory hyperbolic complex manifold (i.e., $H^\infty(M)$ separates points of M). By $G^\infty(M)$ we denote the *geometric predual* to $H^\infty(M)$. By definition, $G^\infty(M)$ is the closed linear span in $H^\infty(M)^*$ of the family $\{\delta_m\}_{m\in M}$ of delta functionals of points in M. We have

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Theorem A

(1) $G^\infty(M)^* = H^\infty(M)$ and any complex Banach space predual to $H^\infty(M)$ is isometrically isomorphic to $G^\infty(M)$.

(2) The closed unit ball $B_{G^\infty(M)}$ of $G^\infty(M)$ is the closed convex hull of the set $\{e^{i\theta}\cdot\delta_m\}_{m\in M,\,\theta\in\mathbb{R}}$.

(3) The injective map $\Delta_M : M \to G^\infty(M)$, $\Delta_M(m) := \delta_m$, is holomorphic with range in the unit sphere of $G^\infty(M)$.

(4) Let Y be a complex Banach space and $H^\infty(M,Y)$ be the space of bounded holomorphic maps $F : M \to Y$ endowed with norm $\|F\| := \sup_{m\in M}\|F(m)\|_Y$. Then the map $I_{M;Y} : \mathcal{L}(G^\infty(M),Y) \to H^\infty(M,Y)$, $I_{M;Y}(T) := T\circ\Delta_M$, is an isometry of complex Banach spaces.

Let us mention other realisations of spaces predual to $H^\infty(M)$. For instance, if $M \Subset \mathbb{C}^n$ is a bounded domain, then the predual space to $H^\infty(M)$ is isometrically isomorphic to $L^1(M)/{}^\perp H^\infty(M)$, where $L^1(M)$ is defined with respect to the Lebesgue measure on $\mathbb{C}^n \cong (\mathbb{R}^{2n})$. If $M := \mathbb{B}^n$ is the open unit ball in $\mathbb{C}^n$, then such predual space is isometrically isomorphic to $L^1(\mathbb{S}^{2n-1})/{}^\perp H^\infty(\mathbb{B}^n)$, where $L^1(\mathbb{S}^{2n-1})$ is defined on the $(2n-1)$-dimensional unit sphere $\mathbb{S}^{2n-1}$ with respect to the $(2n-1)$-Hausdorff measure. (Here for a Banach space X with dual X^* and a subspace $Z \subset X^*$ the space

$$^\perp Z := \{x \in X \,:\, z^*(x) = 0 \quad \forall\, z^* \in Z\}.$$

In the first case this result follows from the fact that $H^\infty(M)$ is a weak* closed subspace of $L^\infty(M)$ while in the second one from the fact that $H^\infty(\mathbb{B}^n)|_{\mathbb{S}^{2n-1}}$ is a weak* closed subspace of $L^\infty(\mathbb{S}^{2n-1})$; here the trace space consists of the boundary values of functions in $H^\infty(\mathbb{B}^n)$.

Let $\Omega \Subset \mathbb{C}^n$ be a starlike domain which admits an exhaustion $\Omega_1 \Subset \Omega_2 \Subset \ldots$, where each Ω_i is open and its closure $\bar{\Omega}_i$ is polynomially convex, i.e., for every $z \notin \bar{\Omega}_i$ there exists a holomorphic polynomial p on $\mathbb{C}^n$ such that $|p(z)| > \max_{\bar{\Omega}_i} |p|$. In particular, the class of such sets Ω contains all bounded convex domains in $\mathbb{C}^n$.

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Theorem 1

$G^\infty(\Omega)$ and $A(\Omega) := H^\infty(\Omega) \cap C(\bar{\Omega})$ have the metric approximation property.

Let M be a Stein manifold, i.e., a complex manifold which admits a proper holomorphic embedding in some $\mathbb{C}^n$ and let $\Omega \Subset M$ be a strongly pseudoconvex domain with C^2 boundary $\partial\Omega$, that is, $\Omega := \{z \in M \,:\, \rho(z) < 0\}$, where ρ is a real-valued C^2 function on a neighbourhood of $\bar{\Omega}$, strongly plurisubharmonic in a neighbourhood of $\partial\Omega$, and $d\rho(z) \neq 0$ for all $z \in \partial\Omega$.

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Let $r : \Omega' \to \Omega$ be an unbranched covering of Ω. It is known that $H^\infty(\Omega')$ separates points of Ω'.

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Let $r : \Omega' \to \Omega$ be an unbranched covering of Ω. It is known that $H^\infty(\Omega')$ separates points of Ω'.

Theorem 2

(a) There exists $\lambda := \lambda(\Omega) \in [1, \infty)$ such that $G^\infty(\Omega')$ has the λ-approximation property.

(b) $A(\Omega) := H^\infty(\Omega) \cap C(\bar{\Omega})$ has the bounded approximation property.

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Proposition 1

(a) If M and N are Caratheodory hyperbolic complex manifolds and $G^\infty(M)$ has the λ-approximation property, then $G^\infty(M \times N)$ is isomorphic (isometrically if $\lambda = 1$) to $G^\infty(N)\hat{\otimes}_\pi G^\infty(M)$ and the Banach-Mazur distance between these spaces is bounded by λ.
If, in addition, $G^\infty(N)$ has the μ-approximation property, then $G^\infty(M \times N)$ has the $\lambda\mu \cdot \min(\lambda, \mu)$-approximation property.

(b) If M and N are bounded domains in Stein manifolds and $A(M)$ has the λ-approximation property, then $A(M \times N)$ is isometrically isomorphic to the injective tensor product $A(M)\hat{\otimes}_\varepsilon A(N)$.
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As an immediate corollary of Theorems 1, 2 and Proposition 1 we obtain:

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Corollary

(a) Let $\Omega_1, \ldots, \Omega_k$ be either starlike domains or unbranched coverings of strongly pseudoconvex domains with C^2 boundaries in Stein manifolds. Then $G^\infty(\Omega_1 \times \cdots \times \Omega_k)$ has the bounded approximation property.

(b) Let $\Omega_1, \ldots, \Omega_k$ be either starlike domains or strongly pseudoconvex domains with C^2 boundaries in Stein manifolds. Then $A(\Omega_1 \times \cdots \times \Omega_k)$ has the bounded approximation property.

Being separable, G^{∞}- and A-spaces of Theorems 1 and 2 are isomorphic to complemented subspaces of separable Banach spaces with bases with the Banach-Mazur distances between them and their isomorphic copies bounded by 4 in the case of Theorem 1 and by a constant depending on Ω in the case of Theorem 2.

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Problem 1

It is an interesting question whether G^∞- and A-spaces of Theorems 1 and 2 have bases. A basis in $A(\mathbb{D})$ was constructed by Bochkarev answering a question of Banach.

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Problem 2

In view of the formualted results it is natural to ask also about existence of domains $\Omega \Subset \mathbb{C}^n$ for which $G^\infty(\Omega)$ and $A(\Omega)$ do not have the approximation property.

Decomposition of G^∞-spaces

Let M be a Caratheodory hyperbolic complex manifold and let $\{M_k\}_{1\le k\le m}$ be an open cover of M. By $i_k : M_k \hookrightarrow M$ we denote the embedding maps. These maps induce linear bounded maps $I_k : G^\infty(M_k) \to G^\infty(M)$ of norm 1 such that $\Delta_M \circ i_k = I_k \circ \Delta_{M_k}$. Consider the direct product $\oplus_{k=1}^m G^\infty(M_k)$ equipped with norm

$$\|(v_1,\dots,v_m)\| := \sum_{k=1}^m \|v_k\|_{G^\infty(M_k)}, \quad (v_1,\dots,v_m)\in \oplus_{k=1}^m G^\infty(M_k).$$

Then there exists a linear bounded map $P : \oplus_{k=1}^m G^\infty(M_k) \to G^\infty(M)$ of norm 1 with dense image defined by the formula

$$P(v_1,\dots,v_m) := \sum_{k=1}^m I_k(v_k), \quad (v_1,\dots,v_m)\in \oplus_{k=1}^m G^\infty(M_k).$$

We say that $G^\infty(M)$ is *decomposable with respect to* $G^\infty(M_1), \ldots, G^\infty(M_m)$ if there exists a linear bounded map $S : G^\infty(M) \to \oplus_{k=1}^m G^\infty(M_k)$ such that $P \circ S = \mathrm{Id}$.

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Next, consider bounded holomorphic maps $s_k : M_k \to \oplus_{k=1}^m G^\infty(M_k)$,

$$s_k(z) := (0, \dots, 0, \underbrace{\Delta_{M_k}(z)}_{k^{\text{th}} \text{ coordinate}}, 0, \dots, 0), \quad z \in M_k.$$

Then the family $\{s_{ij} := s_i - s_j \in H^\infty(M_i \cap M_j, \mathrm{Ker}P)\}$ forms a bounded holomorphic 1-cocycle on the finite open cover $\{M_i\}$ of M with values in $\mathrm{Ker}P$.

We say that such cocyle is H^∞ *trivial* if there exist a finite open refinement $\{M'_j\}$ of $\{M_k\}$ with the refinement map ι from the set of indices of the first cover to that of the second one (i.e., $M'_j \subset M_{\iota(j)}$ for all j) and functions $s'_j \in H^\infty(M'_j, \mathrm{Ker}P)$ such that for all possible p, q with $M'_p \cap M'_q \neq \emptyset$,

$$s'_p(z) - s'_q(z) = s_{\iota(p)\iota(q)}(z) \quad \text{for all} \quad z \in M'_p \cap M'_q.$$

Proposition 2

$G^\infty(M)$ is decomposable with respect to $G^\infty(M_1), \ldots, G^\infty(M_m)$ if and only if the cocycle $\{s_{ij}\}$ is H^∞ trivial.

Theorem 3

Let $r : \Omega' \to \Omega$ be an unbranched covering of a strongly pseudoconvex domain $\Omega \subset \mathbb{C}^p$ with C^2 boundary. Then there exists an open cover $\Omega'_1, \ldots, \Omega'_m$ of Ω' such that each Ω'_i is biholomorphic to direct product $U_i \times \mathbb{N}$, where $U_i \Subset \mathbb{C}^p$ is a convex domain, and $G^\infty(\Omega')$ is decomposable with respect to $G^\infty(\Omega'_1), \ldots, G^\infty(\Omega'_m)$.