Hamilton-Jacobi equation and non-holonomic dynamics

Why use algebroid theory to describe the H-J equation?

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Focus Program on Geometry, Mechanics and Dynamics

and the Legacy of Jerry Marsden

Advances about a formalism which allows to describe Hamilton-Jacobi equation for a great variety of mechanical systems

- Unconstrained systems (Classical hamiltonian systems, reduced hamiltonian systems,.....)
- nonholonomic systems subjected to linear or affine constraints
- dissipative systems subjected to external forces
- time-dependent mechanical systems
- ....

D Iglesias, M de León, D Martín de Diego (2008)

T Ohsawa, A Bloch (2009)

M de León, JC Marrero, D Martín de Diego (2010)

J F Carinena, X Gracia, G Marmo, E Martínez, M C Munoz-Lecanda, N Román-Roy (2010)

P Balseiro, JC Marrero, D Martín de Diego, E P (2010)

M Leok, T Ohsawa, D Sosa (2011)

INGREDIENTS

- Q configuration space (manifold) (q^i)
- $\tau_Q^* : T^*Q \to Q$ phase space of momenta (q^i, p_i)
- $H : T^*Q \to \mathbb{R}$ Hamiltonian function $H(q^i, p_i)$

$$\Downarrow$$

$$X_H \in \mathfrak{X}(T^*Q) \text{ hamiltonian vector field} \qquad X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}$$

- $W : Q \to \mathbb{R}$ the characteristic function $W(q^i)$

Classical Hamilton-Jacobi Theorem

The following sentences are equivalent

1. For every $c : I \to Q$, $c(t) = (q^i(t))$ integral curve of

$$X_H^W = T\tau_Q^* \circ X_H \circ dW \in \mathfrak{X}(Q) \qquad \frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}(q(t), \frac{\partial W}{\partial q}(q(t))$$

$$\Downarrow$$

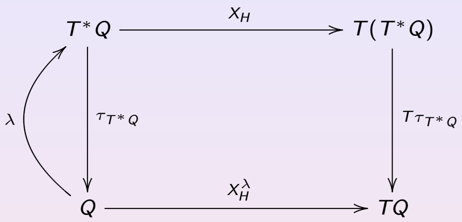
$dW \circ c : I \to T^*Q$ is an integral curve of X_H

2. W satisfies Hamilton-Jacobi equation

$$H \circ dW = constant, \qquad H(q^i, \frac{\partial W}{\partial q^i}) = constant$$

Classical Hamilton-Jacobi equation

Let $\lambda \in \Omega^1(Q)$ be a closed 1-form ($d\lambda = 0$)



Theorem

$$c : I \to Q \text{ integral curve of } X_H^\lambda \Rightarrow \lambda \circ c \text{ integral curve of } X_H,$$

$$\Updownarrow$$

$$X_H \text{ and } X_H^\lambda \text{ are } \lambda\text{-related (i.e. } T\lambda(X_H^\lambda) = X_H).$$

$$\Updownarrow$$

$d(H \circ \lambda) = 0$ Hamilton-Jacobi equation

Tools

$TQ \xrightarrow{\tau_{TQ}} Q$ $\rightsquigarrow$ vector bundle $\tau_D : D \to Q$ over a manifold Q

The canonical symplectic 2-form ω_Q in $T^*Q \simeq$ The canonical Poisson bracket $\{\cdot,\cdot\}_{T^*Q}$ on T^*Q $\rightsquigarrow$ a linear Poisson bracket $\{\cdot,\cdot\}_{D^*}$ on D^*

A Hamiltonian function $H : T^*Q \longrightarrow \mathbb{R}$ $\rightsquigarrow$ a function $H : D^* \to Q$

A section $\lambda : Q \longrightarrow T^*Q$ such that $d\lambda = 0$ $\rightsquigarrow$ a section $\lambda \in \Gamma(D^*)$ which is closed with respect to a *certain differential*

INGREDIENTS

- Q manifold (configuration space)
- D a distribution on Q (constraint distribution)
- g a Riemannian metric on Q
- $V : Q \to \mathbb{R}$ a real function on Q (Potential)

$$\Downarrow$$

$$L : TQ \to \mathbb{R}, \qquad L(v) = \frac{1}{2}g(v,v) - V(\tau(v))$$

$\mathbb{F}L : TQ \to T^*Q$ Legendre transformation

$\mathbb{F}L \equiv$ The vector bundle isomorphism induced by the metric g

$$\Downarrow$$

$\bar{D} = \mathbb{F}L(D)$ the constraint Hamiltonian subbundle of T^*Q

- $H : T^*Q \to \mathbb{R}$ Hamiltonian function

$$\Downarrow$$

$$\bar{X}_H \in \mathfrak{X}(D^*) \quad \bar{X}_H = Ti_D^* \circ X_H \circ P^*$$

$TQ = D \oplus D^\perp \quad P : TQ \to D, \quad P^* : D^* \to T^*Q$

$i_D : D \to TQ, \quad i_D^* : T^*Q \to D^* \quad Ti_D^* : T(T^*Q) \to TD^*$

$$\mathcal{I}(D^0) \equiv \text{the algebraic ideal generated by } D^0$$

Hamilton-Jacobi Theorem for nonholonomic systems

Let $\lambda \in \Omega^1(Q)$ taking values into $\bar{D}$ and satisfying $d\lambda \in \mathcal{I}(D^o)$. Then the following conditions are equivalent:

1. For every integral curve $c : \mathbb{R} \to Q$ of

$$X_H^\lambda = (T\pi_Q) \circ X_H \circ \lambda \in \mathfrak{X}(Q)$$

   then $\lambda \circ c$ is an integral curve of $\bar{X}_H$.

2. $d(H \circ \lambda)(Q) \subset D^o$

D. Iglesias, M. de León, D. Martín de Diego 2008

TOOLS

$D \xrightarrow{\tau_D} Q$ $\rightsquigarrow$ $\tau_D : D \to Q$ vector bundle over a manifold Q

The nonholonomic bracket on D^* $\rightsquigarrow$ an almost linear Poisson $\{\cdot,\cdot\}_{D^*}$ bracket of functions on D^*, i.e., in general, does not satisfy Jacobi identity

$$\{F,G\}_{D^*} = \{F \circ i_D^*, G \circ i_D^*\} \circ P^*, \quad F, G \in C^\infty(D^*)$$

$$TQ = D \oplus D^\perp \quad P : TQ \to D, \quad P^* : D^* \to T^*Q$$

$$i_D : D \to TQ, \quad i_D^* : T^*Q \to D^*$$

A Hamiltonian function $H : T^*Q \longrightarrow \mathbb{R} \Rightarrow \mathcal{H} = H \circ P^* : D^* \longrightarrow \mathbb{R} \rightsquigarrow$ function $\mathcal{H} : D^* \longrightarrow \mathbb{R}$

$$\Downarrow$$

$$\bar{X}_H \in \mathfrak{X}(D^*) \;\; \bar{X}_H(F) = X_{\mathcal{H}}(F) = \{F, \mathcal{H}\}_{D^*}$$

A section $\lambda : Q \longrightarrow T^*Q$ taking values on $\bar{D}$ such that $d\lambda(Q) \subset \mathcal{I}(D^o)$ $\rightsquigarrow$ A section $\lambda \in \Gamma(D^*)$.... and is it closed with respect to a *certain differential operator*?

INGREDIENTS:

- $\tau_D : D \longrightarrow Q$ a vector bundle

$$\Downarrow$$

$$\tau_{D^*} : D^* \longrightarrow Q \text{ its dual vector bundle}$$

- A linear almost Poisson bracket [1] $\{\cdot, \cdot\}_{D^*}$ on D^*

$$\Downarrow$$

$$d^D : \Gamma(\wedge^k D^*) \to \Gamma(\wedge^{k+1} D^*) \text{ differential operator}$$

[1] linear means that the bracket of two linear functions is a linear function

$\tau_D : D \to Q$ vector bundle with linear almost Poisson bracket $\{\cdot,\cdot\}_{D^*}$ on $D^* \to Q$

$\{\hat{X} : D^* \to \mathbb{R} / \hat{X} \text{ is linear}\} \Longleftrightarrow \Gamma(D) = \{X : Q \to D / X \text{ is a section of } \tau\}$

What is the corresponding structure on D?

$\Downarrow$

The bracket of two linear functions with respect to $D^* \to Q$ is again linear

The bracket on the space of sections of D

$[\![\cdot,\cdot]\!]_D : \Gamma(D) \times \Gamma(D) \to \Gamma(D)$ skew-symmetric

$$\widehat{[\![X, Y]\!]_D} = -\{\hat{X}, \hat{Y}\}_{D^*}$$

The bracket of a linear function and a basic function $f \circ \tau_{D^*}$ is a basic function

The vector bundle morphism between D and TQ

$\rho_D : D \to TQ$ (anchor map) $\Rightarrow \rho_D : \Gamma(D) \to \mathfrak{X}(Q)$

$$\rho_D(X)(f) \circ \tau_{D^*} = \{\hat{X}, f \circ \tau_{D^*}\}_{D^*}$$

$$[\![X, fY]\!]_D = f[\![X, Y]\!]_D + \rho_D(X)(f)Y, \quad \forall X, Y \in \Gamma(D), \quad \forall f \in C^\infty(Q)$$

$$\{\{\cdot,\cdot\}_{D^*} \text{ linear almost Poisson bracket on } D^*\}$$

$$\Updownarrow$$

$$\{([\![\cdot,\cdot]\!]_D, \rho_D) \text{ skew-symmetric algebroid structure on } D\}$$

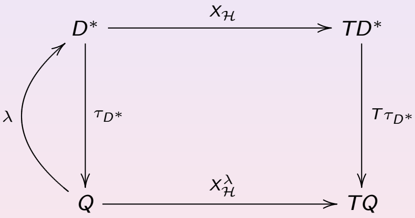
$$d^D : \Gamma(\Lambda^k D^*) \to \Gamma(\Lambda^{k+1} D^*)$$

$$\begin{aligned} d^D\Omega(\xi_0, \xi_1, \dots, \xi_k) &= \sum_{i=0}^{k} (-1)^i \rho_D(\xi_i)(\Omega(\xi_0, \dots, \widetilde{\xi_i}, \dots, \xi_k)) \\ &+ \sum_{i<j} \Omega([\![\xi_i, \xi_j]\!]_D, \xi_0, \dots, \widetilde{\xi_i}, \dots, \widetilde{\xi_j}, \dots, \xi_k) \end{aligned}$$

where $\xi_0, \xi_1, \dots, \xi_k \in \Gamma(D)$

$$\boxed{(d^D)^2 \neq 0}$$

- $\tau_D : D \to Q$ vector bundle
- $\{\cdot,\cdot\}_{D^*}$ linear almost Poisson bracket on D^*
- $\mathcal{H} : D^* \to \mathbb{R}$ Hamiltonian function $\Rightarrow X_{\mathcal{H}} = \{\cdot, H\}_{D^*} \in \mathfrak{X}(D^*)$
- $\lambda : Q \longrightarrow D^*$ be a section of $\tau_{D^*} : D^* \longrightarrow Q$



$$X_{\mathcal{H}}^{\lambda} = T\tau_{D^*} \circ X_{\mathcal{H}} \circ \lambda$$

$$W : Q \to \mathbb{R}, \quad d^D W \text{ is not closed } d^D(d^D W) \neq 0$$

- $\lambda \in \Gamma(D^*)$

$$\Upsilon^\lambda : \Omega^1(D^*) \to \Gamma(D), \quad \eta(\Upsilon^\lambda(\beta)) = \beta(\eta^v) \circ \lambda \quad \beta \in \Omega^1(D^*), \eta \in \Gamma(D^*)$$

$$\eta^v \in \mathfrak{X}(D^*)$$

$$\delta^\lambda_{\mathcal{H}} \in \Gamma(D) = \Upsilon^\lambda(d\mathcal{H})$$

Hamilton-Jacobi Theorem

$c : I \to Q$ integral curve of $X^\lambda_{\mathcal{H}} \in \mathfrak{X}(Q) \Rightarrow \lambda \circ c$ integral curve of $X_{\mathcal{H}} \in \Gamma(D^*)$

$$\Updownarrow$$

$$i_{\delta^\lambda_{\mathcal{H}}} d^D \lambda + d^D(\mathcal{H} \circ \lambda) = 0$$

M DE LEÓN, JC MARRERO, D MARTÍN DE DIEGO (2010)

The general distribution $\widetilde{D} = \rho_D(D)$ *bracket generating*

$$\Updownarrow$$

$\{X_k, [X_k, X_l], [X_i, [X_k, X_l]..../X_j \in \widetilde{D}\}$ spans $\mathfrak{X}(Q)$

$\mathrm{Lie}^\infty(\widetilde{D})$ the smallest Lie subalgebra of $\mathfrak{X}(Q)$ containing $\widetilde{D}$

$$d^D(\mathcal{H} \circ \lambda) = 0$$

$$\Updownarrow$$

$\mathcal{H} \circ \lambda$ is constant on the leaves of the foliation $\mathrm{Lie}^\infty(\widetilde{D})$

- $\mathfrak{g} \to \{x\}$ with Lie-Poisson structure on $\mathfrak{g}^*$. Thus, if $D = \mathfrak{h}$ is a subspace of $\mathfrak{g}$, we obtain that the nonholonomic bracket (nonholonomic Lie-Poisson bracket)
- A principal G-bundle $\pi : Q \to Q/G$

$$\tau_{TQ} : TQ \to Q \text{ is equivariant}$$

$$\Downarrow$$

$$TQ/G \to Q/G$$

The linear Poisson structure on $(T^*Q)/G$ is characterized by the following condition: the canonical projection $T^*Q \to T^*Q/G$ is a Poisson epimorphism

the Hamilton-Poincare bracket on T^*Q/G

D a G-invariant distribution on Q

$$\Downarrow$$

$$D/G \text{ is a vector subbundle of } TQ/G$$

$$\Downarrow$$

the non-holonomic Hamilton-Poincaré bracket on D^*/G

INGREDIENTS

- $\pi : Q \to \mathbb{R}$ fibration (configuration space) $\quad \pi(q^i, t) \to t$

$$\eta = \pi^*(dt) \in \Omega^1(Q) \qquad \eta = dt$$

- phase space of momenta
  - extended T^*Q
  - restricted $V^*\pi \quad V\pi = \{v \in TQ / \eta(v) = 1\}$

$$\Downarrow$$

Principal $\mathbb{R}$-bundle $\mu : T^*Q \to V^*\pi \qquad \mu(q^i, t, p_i, p_t) \to (q^i, p_i, t)$

- $h : V^*\pi \to T^*Q$ Hamiltonian section of $\mu \qquad h(q^i p_i, t) \to (q^i, p_i, t, -H(q^i, p_i, t))$

$$F_h : T^*Q \to \mathbb{R} \qquad F_h(q^i, t, p_i, p_t) \to H(q^i, p_i, t) + p_t$$

$$\mu(\alpha - h\mu(\alpha)) = 0 \Longrightarrow \alpha - h\mu(\alpha) = F_h(\alpha)\eta$$

$$R_h \in \mathfrak{X}(V^*\pi) \qquad R_h(F) \circ \mu = \{F \circ \mu, F_h\} \qquad R_h = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}$$

- $W : Q \to \mathbb{R}$ the characteristic function $\quad W(q^i, t)$

$$Q = M \times \mathbb{R}$$

$$h : V^* pi = T^* M \times \mathbb{R} \to T^* Q = T^*(M \times \mathbb{R}), \quad h(q^i, p_i, t) \to (q^i, t, p_i, -H(q^i, p_i, t))$$

$$H : V^* \pi : T^* M \times \mathbb{R} \to \mathbb{R}$$

Hamilton-Jacobi Theorem for time-dependent Mechanics

The following sentences are equivalent

1. For every curve $c : I \to Q$ such that

$$c'(t) = T\tau_Q^* \circ X_{H_t}(dW_t(c(t)))$$

$$\Downarrow$$

$dW \circ c : I \to T^* Q$ is an integral curve of X_H.

2. W satisfies Hamilton-Jacobi equation

$$H_t \circ dW_t + \frac{\partial W}{\partial t} = constant$$

Tools

$\tau_Q : TQ \longrightarrow Q$,$\rightsquigarrow$ $\tau_D : D \to Q$ a vector bundle with a almost linear Poisson bracket $\{\cdot,\cdot\}_{D*}$

$\eta \in \Gamma(T^*Q)$ such that $d\eta = 0$ and $\eta(q) \neq 0 \quad \forall q \in Q$ $\rightsquigarrow$ a section $\phi : Q \to D^*$ not null in everywhere such that $d^D\phi = 0$

$$\Downarrow$$

$$\hat{\eta} : TQ \to \mathbb{R} \text{ linear function} \rightsquigarrow \hat{\phi} : D \to \mathbb{R} \quad \text{linear function}$$

$$\hat{\eta}^{-1}(0) = V\pi \quad \mu : T^*Q \to (\hat{\eta}^{-1}(0))^* \rightsquigarrow \hat{\phi}^{-1}(0) = V \quad \mu : D^* \to V^*$$

$\{\cdot,\cdot\}_V^*$ linear almost Poisson braket such that μ an almost Poisson morphim

A hamiltonian section $h : (\widehat{\phi}^{-1}(0))^* \longrightarrow T^*Q$ $\rightsquigarrow$ A section $h : V^* \to D^*$ of μ

$$F_h : D^* \to \mathbb{R}$$

A section $\lambda : Q \longrightarrow T^*Q$ such that $d\lambda = 0$ $\rightsquigarrow$ A section $\lambda : Q \to D^*$ of D^*

$$\Upsilon^{\lambda} : \Omega^1(D^*) \to \Gamma(D), \quad \delta_H^{\lambda} = \Upsilon^{\lambda}(dF_h) \in \Gamma(D)$$

Hamilton-Jacobi Theorem

$$c : I \to Q \text{ integral curve of } R_h^{\lambda} = T\tau_{V^*} \circ R_h \circ \mu \circ \lambda \in \mathfrak{X}(Q)$$

$$\Rightarrow \mu \circ \lambda \circ c \text{ integral curve of } R_h \in \mathfrak{X}(V^*)$$

$$\Updownarrow$$

$$\mu \circ i_{\delta_h}^{\lambda} d^D \lambda + d^V (F_h \circ \lambda) = 0$$

- **The vector bundle**: $TQ \times \mathbb{R} \to Q$
- **The linear almost Poisson bracket:**

  $\mathcal{F} : TQ \to TQ$ vector bundle morphism$\equiv \beta \in \Omega^1(TQ)$ semibasic homogeneous of degree 1

$$\Pi_{T^*Q\times\mathbb{R}} = \Pi_{T^*Q} + \frac{\partial}{\partial t} \wedge Y_{\mathcal{F}}$$

$$Y_{\mathcal{F}} \in \mathfrak{X}(T^*Q) \quad Y_{\mathcal{F}}(\alpha) = \mathcal{F}^*(\alpha)^v_\alpha \in T_\alpha(T^*Q)$$

- $R_h = X_H - Y_{\mathcal{F}} \in \mathfrak{X}(T^*Q)$
- **The** 1**-cocyple** $\phi = (0,1) \in \Gamma(T^*Q \times \mathbb{R}) \cong C^\infty(Q) \times \mathfrak{X}(Q) \Rightarrow V = TQ$
- $\mu = p_1 : T^*Q \times \mathbb{R} \to T^*Q$
- **The Hamiltonian section:**
  $H : T^*Q \to \mathbb{R} \Rightarrow h : T^*Q \to T^*Q \times \mathbb{R}, \quad h(\beta) = (\beta, -H(\beta))$

$$\Upsilon^{\lambda} : \Omega^1(T^*Q) \to \mathfrak{X}(Q), \quad \delta_H^{\lambda} = \Upsilon(dH)$$

Hamilton-Jacobi Theorem

$\lambda \in \Omega^1(Q)$

$$c : I \to Q \text{ integral curve of } R_h^{\lambda} = T\tau_{T^*Q} \circ X_H \circ \lambda \in \mathfrak{X}(Q)$$

$$\Rightarrow \lambda \circ c \text{ integral curve of } X_H - Y_{\mathcal{F}} \in \mathfrak{X}(T^*Q)$$

$$\Updownarrow$$

$$i_{\delta_H^{\lambda}} d\lambda + d(H \circ \lambda) + Y_{\mathcal{F}}(\lambda) = 0$$

INGREDIENTS

- a vector subbundle $\tau : U \to Q$ of $(\tau_D : D \to Q, \{\cdot, \cdot\}_{D^*})$
- a bundle metric $\mathcal{G} : D \times_Q D \to \mathbb{R} \Rightarrow P : D = U \oplus U^{\perp} \to U$
- a function $V : Q \to \mathbb{R}$
- $X_0 \in \Gamma(D)$ such that $P(X_0) = 0$

$$\Downarrow$$

affine nonholonomic constraints $\equiv \tau_{\mathcal{U}} : \mathcal{U} \to Q$

$$q \in Q \longrightarrow \mathcal{U}_q = \{u_q + X_0(q) / u_q \in U_q\}$$

- **The vector bundle** $\tau_{\widetilde{\mathcal{U}}} : \widetilde{\mathcal{U}} = (\mathcal{U}^+)^* \to Q$ (it is a subbundle of $D \times \mathbb{R} \to Q$)

$$\Gamma(\widetilde{\mathcal{U}}) \equiv < \{(\sigma + fX_0, f)/\sigma \in \Gamma(U), \quad f \in C^\infty(Q)\} >$$

- **The linear almost Poisson manifold on** $\widetilde{\mathcal{U}}^* \cong \mathcal{U}^+$

$$\Updownarrow$$

$([\![\cdot,\cdot]\!]_D, \rho_D)$ skewsymmetric algebroid

$$\widetilde{P} : D \times \mathbb{R} \to \widetilde{\mathcal{U}}, \quad \widetilde{P}(e_q, s) = (P(e_q) + sX_0(q), s)$$
$$P : D = U \oplus U^\perp \to U$$

$$\begin{aligned}[\![(\sigma_1 + f_1X_0, f_1), (\sigma_2 + f_2X_0, f_2)]\!]_{\widetilde{\mathcal{U}}} \quad = \quad & \widetilde{P}([\![\sigma_1 + f_1X_0, \sigma_2 + f_2X_0]\!]_D, \\ & \rho_D(\sigma_1 + f_1X_0)(f_2) - \rho_D(\sigma_2 + f_2X_0)(f_1))\end{aligned}$$

$$\rho_{\widetilde{\mathcal{U}}}(\sigma + fX_0, f) = \rho_D(\sigma + fX_0)$$

- **The 1-cocycle** $\phi \in \Gamma(\widetilde{\mathcal{U}}^*)$

$$\phi : \Gamma(\widetilde{\mathcal{U}}) \to C^\infty(Q) \quad \phi(\sigma + fX_0, f) = f$$

The Hamilton-Jacobi equation of a mechanical system subjected to affine nonholonomic constraints

$$V = U, \quad ([\![\cdot,\cdot]\!]_U = P \circ [\![\cdot,\cdot]\!]_D, \quad \rho_U = \rho)$$

the Hamiltonian section $h : U^* \to \widetilde{\mathcal{U}}^*$

$$H : U^* \to \mathbb{R} \quad H(\alpha) = \frac{1}{2}\mathcal{G}_{U^*}(\alpha,\alpha) + V(q)$$

$$h(\gamma) = (u_q + sX_0(q), s) = \gamma_q(u_q) - sH(\gamma)$$

$$\Upsilon^\lambda : \Omega^1(U^*) \to \Gamma(U) \quad \delta_H^\lambda = \Upsilon^\lambda(dH) \in \Gamma(U)$$

Hamilton-Jacobi Theorem

Assume that $\lambda \in \Gamma(U^*)$

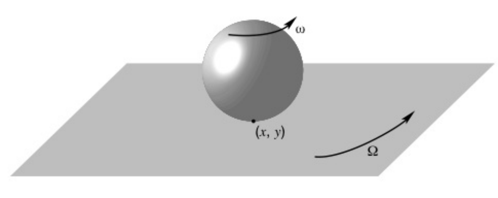
$c : I \to Q$ integral curve of $R_h^\lambda = T\tau_{U^*} \circ R_h \circ \lambda \in \mathfrak{X}(Q)$

$\Rightarrow \lambda \circ c$ is a solution of Hamilton equations

$$\Updownarrow$$

$$i_{\delta_H^\lambda} d^U \lambda + \mu \circ i_{(X_0,1)} d^{\widetilde{\mathcal{U}}}(h \circ \alpha) = 0$$

We consider a homogeneous ball with radius $r > 0$, mass m and inertia mk^2 about any axis. Suppose that the ball rolls without sliding on a horizontal table which rotes with a time-dependent angular velocity $\Omega(t)$ about vertical axis thought of one of its point. Apart from the gravitational force, no other external forces are assumed.



Configuration space: Choose a cartesian reference frame with origin at the center of rotation of the table and $z-$axis along the rotation axis. (q_1, q_2)=the position of the point of contact of the sphere with the table.

$$(t, q_1, q_2) \in Q := \mathbb{R}^3$$

An example: An homogeneous rolling ball without sliding on a rotating table with time-dependent angular velocity

$$(t, q^1 q^2, \dot{q}^1, \dot{q}^2, \omega_1, \omega_2, \omega_3) \in \mathbb{R} \times T\mathbb{R}^2 \times \mathbb{R}^3$$

ω_1, ω_2 and ω_3 are the components of the angular velocity of the sphere

- **The extended phase space of momenta:** $T^*\mathbb{R}^3 \times \mathbb{R}^3$
- **The restricted phase space of momenta:** $\mathbb{R} \times T^*\mathbb{R}^2 \times \mathbb{R}^3$

$$\mu : T^*\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R} \times T^*\mathbb{R}^2 \times \mathbb{R}^3$$

The hamiltonian section $h : \mathbb{R} \times T^*\mathbb{R}^2 \times \mathbb{R}^3 \to T^*\mathbb{R}^3 \times \mathbb{R}^3$

$$h(t, q^i, p_i, \pi_i) = (t, q^i, -H(t, q^i, p_i, \pi_i), p_i, \pi_i)$$

$$H = \frac{1}{2}\left(\frac{1}{m}(p_1^2 + p_2^2) + \frac{1}{mk^2}(\pi_1^2 + \pi_2^2 + p_2^2)\right)$$

Ball without sliding

$\Downarrow$

The affine constraints

$$\begin{aligned} \dot{q}_1 - r\omega_2 &= -\Omega(t) q_2 \\ \dot{q}_2 + r\omega_1 &= \Omega(t) q_1 \end{aligned}$$

$$\begin{aligned} \Omega(t) q^2 + \tfrac{1}{m} p_1 - \tfrac{r}{mk^2} \pi_2 &= 0 \\ -\Omega(t) q^1 + \tfrac{1}{m} p_2 - \tfrac{r}{mk^2} \pi_1 &= 0 \end{aligned}$$

Hamilton equations

$$\begin{aligned}
\dot{q}^1 &= \frac{1}{m}p_1 \\
\dot{q}^2 &= \frac{1}{m}p_2 \\
\dot{p}_1 &= -\frac{mk^2}{k^2+r^2}\left(\frac{d\Omega(t)}{dt}q^2 + \Omega(t)\frac{p_2}{m}\right) \\
\dot{p}_2 &= \frac{mk^2}{k^2+r^2}\left(\frac{d\Omega(t)}{dt}q^1 + \Omega(t)\frac{p_1}{m}\right) \\
\dot{\pi}_1 &= \frac{rmk^2}{k^2+r^2}\left(\frac{d\Omega(t)}{dt}q^1 + \Omega(t)\frac{p_1}{m}\right) \\
\dot{\pi}_2 &= \frac{rmk^2}{k^2+r^2}\left(\frac{d\Omega(t)}{dt}q^2 + \Omega(t)\frac{p_2}{m}\right) \\
\dot{p}_3 &= 0
\end{aligned}$$

The vector bundle: $\tau : D = T\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$

Global basis of $\Gamma(T\mathbb{R}^3 \times \mathbb{R}^3)$

$$e_0 = (\frac{\partial}{\partial t} - \Omega(t)q^2\frac{\partial}{\partial q^1} + \Omega(t)q^1\frac{\partial}{\partial q^2}, 0), \quad e_1 = (\frac{\partial}{\partial q^1}, 0), \quad e_2 = (\frac{\partial}{\partial q^2}, 0),$$
$$e_3 = (0, (1, 0, 0)), \quad e_4 = (0, (0, 1, 0)), \quad e_5 = (0, (0, 0, 1)),$$

The linear almost Poisson structure on $D^* = T^*\mathbb{R}^3 \times \mathbb{R}^3$

$$[\![e_0, e_1]\!]_D = -\Omega(t)e_2, \quad [\![e_0, e_2]\!]_D = \Omega(t)e_1, \quad [\![e_3, e_4]\!]_D = e_5,$$

$$[\![e_4, e_5]\!]_D = e_3, \quad [\![e_5, e_3]\!]_D = e_4,$$

$$\rho_D(e_0) = \frac{\partial}{\partial t} - \Omega(t)q^2\frac{\partial}{\partial q^1} + \Omega(t)q^1\frac{\partial}{\partial q^2}, \quad \rho_D(e_1) = \frac{\partial}{\partial q^1}, \quad \rho_D(e_2) = \frac{\partial}{\partial q^2}.$$

Subbundle of D

$$U := span\{e_3 - re_2 \, , \, e_4 + re_1 \, , \, e_5\}$$

Fiber metric on D

$$\mathcal{G} = {e_0}^2 + (m((e_1)^2 + (e_2)^2) + mk^2((e_3)^2 + (e_4)^2 + (e_5)^2)$$

The section X_0 of D

$$X_0 = e_0$$

The section λ of U^*

$$\lambda = d^U(\varphi_1(t)q^1 + \varphi_2(t)q^2)$$

$$d^U\lambda \neq 0$$

If $\Omega(t) = \Omega_0 t$

Solution of Hamilton equations

$$\lambda \circ c(t) = (t, q^1(t), q^2(t); \lambda_3(c(t)), \lambda_4(c(t)), 0)$$

$$\begin{aligned}
\lambda_3(c(t)) &= \frac{-r}{\sqrt{m(k^2+r^2)}}\left(C_1 \sin\left(\frac{r^2\Omega_0 t^2}{2(k^2+r^2)}\right) + C_2 \cos\left(\frac{r^2\Omega_0 t^2}{2(k^2+r^2)}\right)\right), \\
\lambda_4(c(t)) &= \frac{r}{\sqrt{m(k^2+r^2)}}\left(C_1 \cos\left(\frac{r^2\Omega_0 t^2}{2(k^2+r^2)}\right) - C_2 \sin(\frac{r^2\Omega_0 t^2}{2(k^2+r^2)})\right),
\end{aligned}$$

where C_1, C_2 are real constants.

Using the linear almost Poisson theory (or skew-symmetric algebroid theory) we have given a simple method to describe the Hamilton-Jacobi equations for several situations. Usually, these equations make it easy to find solutions for the equations of Hamilton equations.

Thanks for your attention!