Calegari - "Modular Forms" 3/12/12

§ Lattices and Elliptic Curves

<u>Def'n</u>: A lattice $\Lambda \subseteq \mathbb{C}$ consists of a discrete embedding $\mathbb{Z}^2 \longrightarrow \mathbb{C}$

So we have $\Lambda = \{\omega_1, \omega_2\} = \{\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2\}$ where $\omega_1, \omega_2 \neq 0$ and $\omega_1/\omega_2 \notin \mathbb{R}$

Note that Λ does not determine a unique basis. In fact, we have the equivalence relation: $\{\omega_1, \omega_2\} = \{a\omega_1 + b\omega_2, c\omega_1 + d\omega_2\}$ if and only if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})$.

<u>Def'n</u>: 2 lattices Λ and Λ' are <u>homothetic</u> if $\exists \mu \in \mathbb{C}^\times$, $\Lambda' = \mu\Lambda$

<u>Thm 1</u>: $\mathbb{C}/\Lambda$ is an algebraic variety.

To find a function on the lattice, first try $\sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^2}$, which has convergence issues so salvage to get $\left(\sum_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right) + \frac{1}{z^2} =: X$

We can also define $y = \frac{dx}{dz} = -\frac{2}{z^3} + \sum_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \frac{-2}{(z-\lambda)^3}$

Using the classical ~~functions~~ constants G_4, G_6 (defined later), we can see

$$y^2 - 4x^3 - 60G_4x - 140G_6 = O(z) \quad \text{as } z \longmapsto 0$$

This is holomorphic + bounded $\Longrightarrow$ constant $\Longrightarrow 0$

<u>Lemma</u>: $\mathbb{C}/\Lambda \cong \mathbb{C}/\Lambda'$ iff Λ and Λ' are homothetic.
(isomorphic as alg varieties)

Pf: We have

$$\begin{tikzcd} \mathbb{C} \arrow[r, dashed, "\tilde{f}"] \arrow[d] \arrow[dr] & \mathbb{C} \arrow[d] \\ \mathbb{C}/\Lambda \arrow[r, "\sim", "f"'] & \mathbb{C}/\Lambda' \end{tikzcd}$$

holomorphic

$\tilde{f}$ is a lift of f and is holomorphic.

We know $f(\Lambda) \subseteq \Lambda'$

For $\lambda \in \Lambda$, $\tilde{f}(x+\lambda) - \tilde{f}(x) \in \Lambda'$ $= \text{constant}$.

so $\tilde{f}'(x+\lambda) - \tilde{f}'(x) = 0$

hence $\tilde{f}'(x) = \mu x$

$\Rightarrow \Lambda' = \mu\Lambda$

Can work backwards (exercise).

Elliptic curves $\longleftrightarrow$ lattices up to homothety

given $\Lambda = \{\omega_1, \omega_2\} \xleftrightarrow{\text{up to homothety}} \{\tau, 1\}$ s.t. $\operatorname{Im}(\tau) \neq 0$

$\{a\omega_1 + b\omega_2, c\omega_1 + d\omega_2\} \longleftrightarrow \{\frac{a\tau+b}{c\tau+d}, 1\}$ $\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})$

so elliptic curves $\longleftrightarrow \{\tau \in \mathbb{C}, \operatorname{Im}(\tau) \neq 0\} / GL_2(\mathbb{Z})$

let $\mathbb{H} = \{\tau \in \mathbb{C} : \operatorname{Im}(\tau) > 0\}$

Then $\{\tau \in \mathbb{C}, \operatorname{Im}(\tau) \neq 0\} / GL_2(\mathbb{Z}) \longleftrightarrow \mathbb{H} / SL_2(\mathbb{Z})$

§ Modular Forms

__Def'n:__ A modular form of weight k ($\in \mathbb{Z}$) is a holomorphic function on $\mathbb{H}$
s.t. $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$
and is "bounded at ∞" i.e. $\lim_{\tau \to i\infty} |f(\tau)| < \infty$

__Def'n 2:__ A modular form of wt k is a (holomorphic) function on lattices
s.t. $f(\mu\Lambda) = \mu^{-k} f(\Lambda)$ and "bounded at $i\infty$", i.e. $\lim_{\tau \to i\infty} |f(\{\tau, 1\})| < \infty$

__Def'n 3:__ (Something in terms of elliptic curves)
In order to do so, we must refine our equivalence EC $\longleftrightarrow$ lattices / homothety

$\mathbb{C}$ has a differential dz. This descends to a differential on $\mathbb{C}/\Lambda$
$\Lambda \longmapsto \mathbb{C}/\Lambda$, $dz \in H^0(\mathbb{C}/\Lambda, \Omega^1) \cong \mathbb{C}$ (generated by dz)
Say we have $\eta_1, \eta_2 \in H^0(\mathbb{C}/\Lambda, \Omega^1)$. Then $\frac{\eta_1}{\eta_2} = f$ on $\mathbb{C}/\Lambda$

Any $\eta \in H^0(\mathbb{C}/\Lambda, \Omega^1)$ is of the form $\eta = f\,dz$

What does this mean for two lattices equivalent up to homothety?

Say we have $\mathbb{C}/\Lambda$ $\qquad$ $\mathbb{C}/\mu\Lambda$

call differential "dz" $\qquad$ "dz'"

$\boxed{\text{Recall } X \simeq \sum_{\mu\Lambda}' \frac{1}{(\mu z - \mu\lambda)^2}}$

$y^2 = 4x^3 - ax + b$ $\qquad$ $Y^2 = 4X^3 - AX - B$

then $X = \frac{x}{\mu^2}$ $\quad$ $Y = \frac{y}{\mu^3}$

$A = \frac{a}{\mu^4}$ $\quad$ $B = \frac{b}{\mu^6}$

$\left.\begin{array}{l} \frac{dx}{y} = dz \\ \text{or equivalently} \\ \frac{dx}{dz} = y \end{array}\right\} \mathbb{C}/\Lambda$

for $\mathbb{C}/\Lambda'$, we find $\frac{dX}{Y} = \mu^{\pm 1}\frac{dx}{y}$

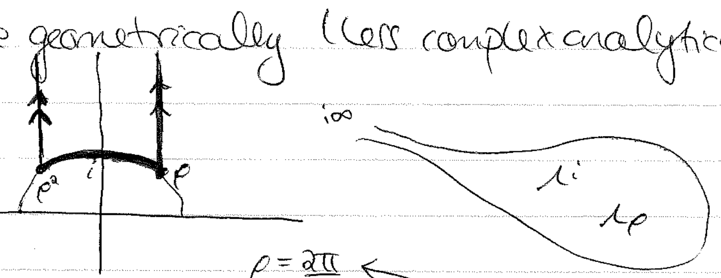
$(\Lambda' = \mu\Lambda)$

<u>Def'n 3</u>: A modular form of wt k is a function on pairs (E, ω) where $\omega \in H^0(E, \Omega^1)$ s.t. $f(E, \mu\omega) = f(E, \omega)\mu^k$ + "bounded at $i\infty$."

Now, we want to think more geometrically (less complex analytically)

$\mathbb{H}/SL_2(\mathbb{Z})$ looks like

(or rather $SL_2(\mathbb{Z})\backslash\mathbb{H}$)



$\rho = \frac{2\pi}{3}$ ← identified w/ ρ^2

$i = \frac{2\pi}{2}$

Everything we have said may be refined to deal with "Level structure"

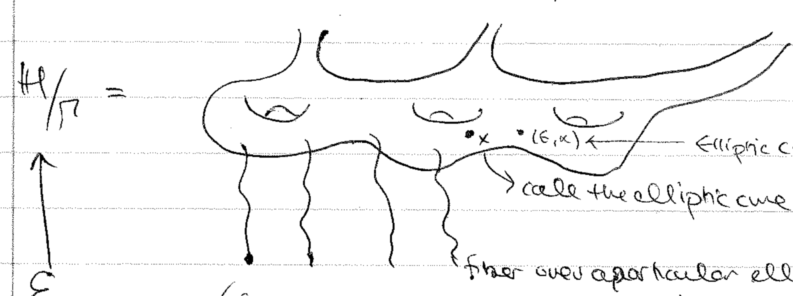
$\Lambda \longmapsto \Lambda, P$ $\quad$ where P is s.t. $NP \in \Lambda$

Λ w/ level structure $\longrightarrow \{\tau, 1\}$ where $\tau \in \mathbb{H}/\Gamma$ where $\Gamma \leq SL_2(\mathbb{Z})$

$\cup\!\!\mid$

$\Gamma(N) = \ker(\Gamma \to SL_2(\mathbb{Z}/N\mathbb{Z}))$

$\mathbb{H}/\Gamma =$



← Elliptic curve E, level structure α

call the elliptic curve E_x, $H^0(E_x, \Omega^1)$

fiber over a particular elliptic curve is all points x s.t. $E_x = E$

(forgetting the level structure)

We have a section $\mathcal{E}$ $\xrightarrow{\pi}$ $Y(\Gamma)$ $\quad \omega := \pi_* \Omega^1_{\mathcal{E}/Y(N)}$ sheaf

$\omega_x = H^0(\mathcal{E}_x, \Omega^1)$ at a point x on $Y(\Gamma)$

Then modular forms of weight k "=" sections of $H^0(Y(\Gamma), \omega^{\otimes k})$
(almost.)

(*) also have issue of dimensionality w/ sections of $Y(\Gamma)$

We need to deal w/ the boundedness condition at $i\infty$

→ (*) This is solved by compactifying $Y(\Gamma) \longrightarrow X(\Gamma) = \overline{Y(\Gamma)}$

(*) so then modular forms of weight k are __exactly__ sections of $H^0(X(\Gamma), \omega^{\otimes k})$

$\omega^{\otimes 2}$ is very ample

Furthermore, dim of $H^0(X(\Gamma), \omega^{\otimes k})$ can be given by Rieman-Roch if $k \geq 2$.

For $k=2$, we have $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 f(\tau)$ $\qquad d\left(\frac{a\tau+b}{c\tau+d}\right) = \frac{d\tau}{(c\tau+d)^2}$

$f(\tau)d\tau$ is defined on $\mathbb{H}$, invariant under $\Gamma = SL_2(\mathbb{Z})$

modular form of wt $= 2 \overset{?}{=\!=} H^0(X(\Gamma), \Omega^1)$

let $\tau \longrightarrow i\infty$, write $q = e^{2\pi i \tau}$. Then $\frac{dq}{2\pi i q} = d\tau$ so $f(\tau)d\tau$ can be written as $\frac{f(q)}{2\pi i} \cdot \frac{dq}{q}$

So we allow a simple pole at the cusps.

Then we can write modular form of wt $2 = H^0(X(\Gamma), \Omega^1(\text{cusps}))$

__Note__: there is an isomorphism $\omega^{\otimes 2} \cong \Omega^1(\text{cusps})$

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§ __Cohomology__ (still $k=2$)

$H^*_{sing}(X(\Gamma), \mathbb{C}) \cong H^*_{dR}(X(\Gamma), \mathbb{C})$

For curves, there is an exact sequence

$$1 \longrightarrow \underbrace{H^0(X, \Omega^1)}_{k=2} \longrightarrow H_{dR} \longrightarrow \underset{\substack{\text{\\|\\|} \\ H^0(X,\Omega^1)^\vee}}{H^1(X, \mathcal{O}_X)} \longrightarrow 1$$

by Serre duality

ω

can represent class in here by a harmonic form (satisfying explicit differential equation)

for curves, essentially satisfying Cauchy-Riemann eqns (or conjugate)

so holomorphic forms or anti-holomorphic forms
dz or $d\bar{z}$

For more general MF of wt k.

$\longleftrightarrow H^1(X(\Gamma), \underbrace{\mathrm{Sym}^{k-2}\,\mathbb{C}^2}_{\text{local system}})$

only works if $k \geq 2$