On the flexibility of Kokotsakis meshes

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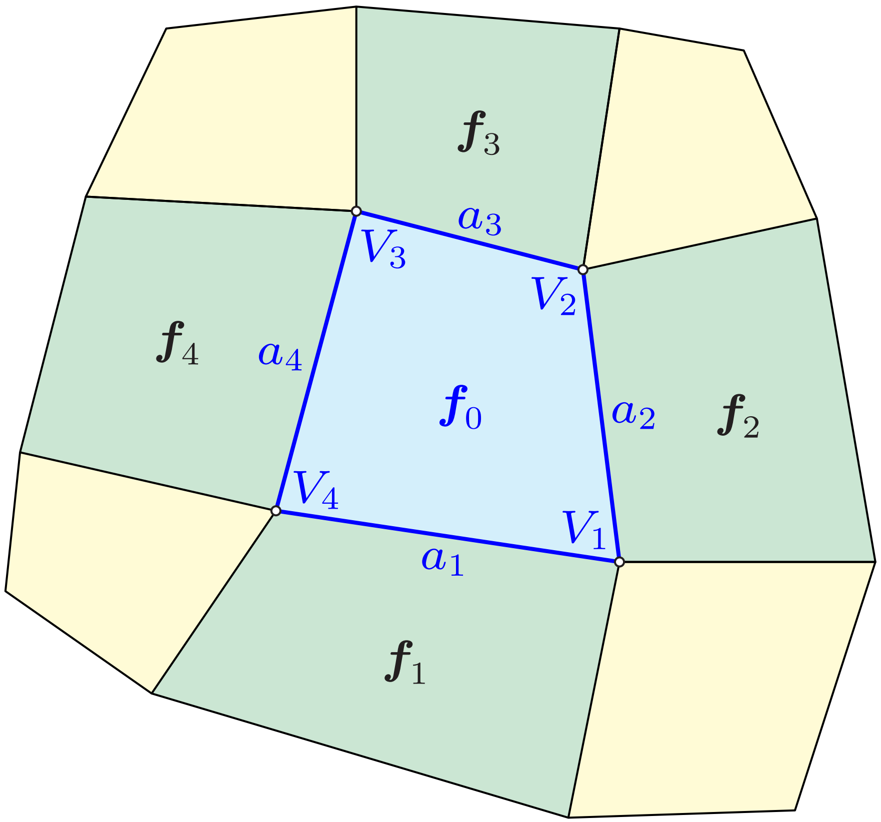
Outline

1. Definition of Kokotsakis meshes
2. Three examples of flexible quad meshes
3. Transmission by one spherical four-bar
4. Composition of spherical four-bars
5. Flexibility vs. reducibility of meshes

1. Definition of Kokotsakis meshes



Special case $n = 4$

A Kokotsakis mesh is a polyhedral structure consisting of an n-sided central polygon $\boldsymbol{f}_0$ surrounded by a belt of polygons.

Each side a_i , $i = 1, \ldots, n$, of $\boldsymbol{f}_0$ is shared by a polygon $\boldsymbol{f}_i$. Each vertex V_i of $\boldsymbol{f}_0$ is the meeting point of four faces.

Each face is seen as a rigid body; only the dihedral angles can vary. Under which conditions a Kokotsakis mesh is continuously flexible ?

1. Definition of Kokotsakis meshes

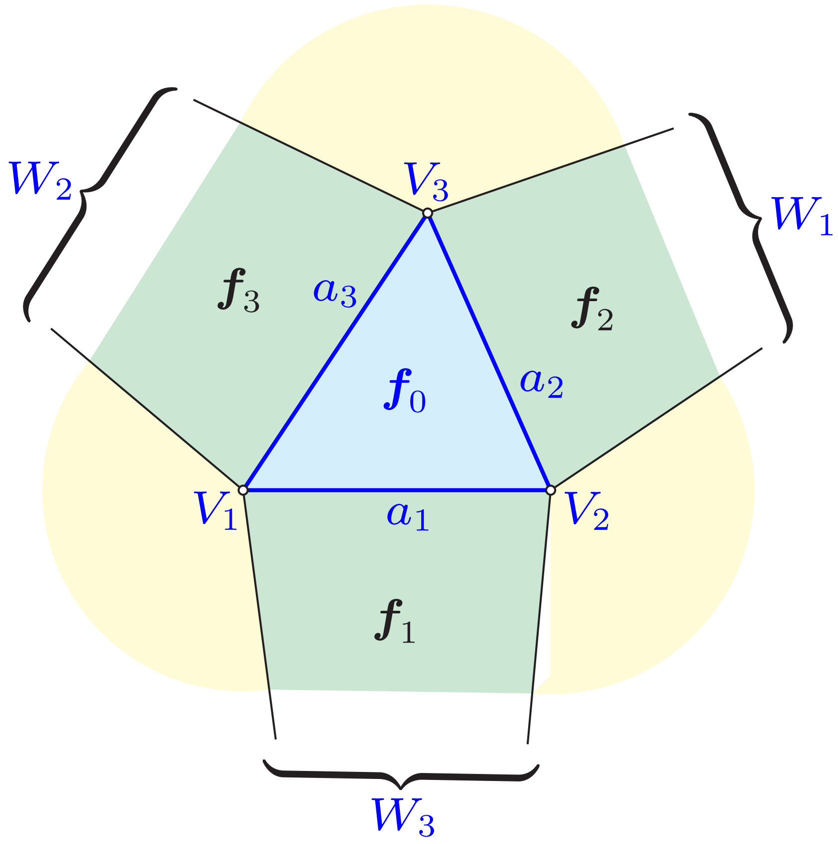
Antonios KOKOTSAKIS
1899–1964

He was born on the island Crete in Greece. As a precocious child, he was accepted at the Department of Civil Engineering of Technical University of Athens already in the age of 16 .

After graduation he was appointed a lecturer in the Department of Descriptive and Projective Geometry. He finished his PhD-thesis entitled *"About flexible polyhedra"* under the supervision of K. CARATHEODORI in Munich/Germany.

His list of publications contains not more than 5 titles.

1. Definition of Kokotsakis meshes



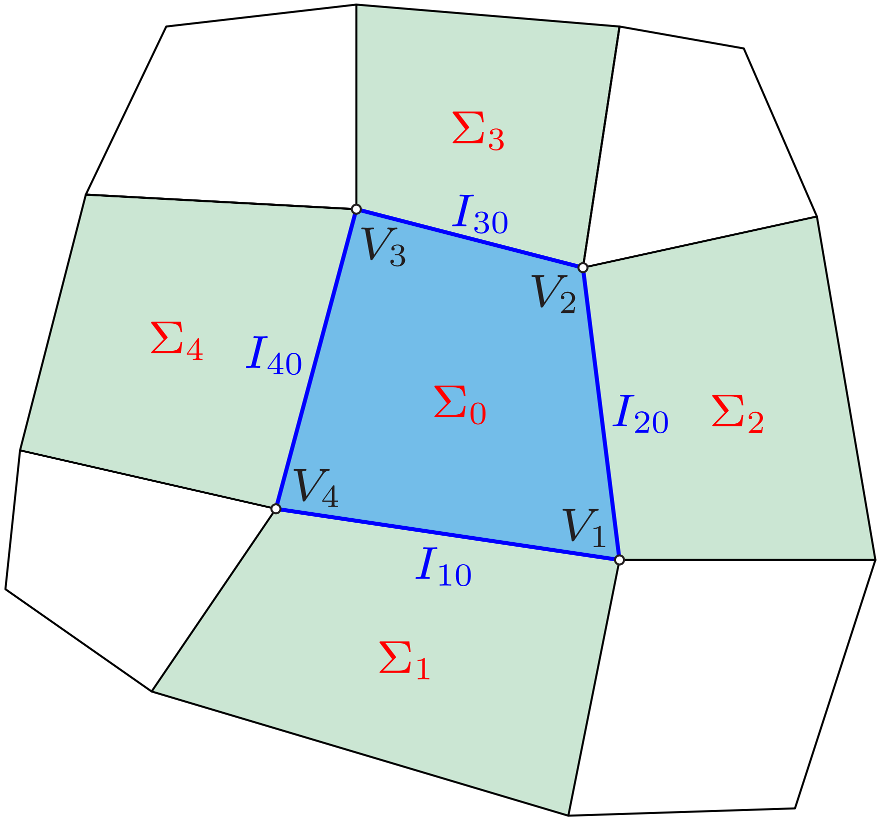
Special case: $n = 3$

A Kokotsakis mesh for $n = 4$ is also called Neunflach [German] (nine-flat) (Kokotsakis 1931, Sauer 1932)

For $n = 3$ the Kokotsakis mesh is equivalent to an octahedron with $V_1V_2V_3$ and $W_1W_2W_3$ as opposite triangular faces.

This offers an alternative approach to R. Bricard's *flexible octahedra*.

1. Definition of Kokotsakis meshes



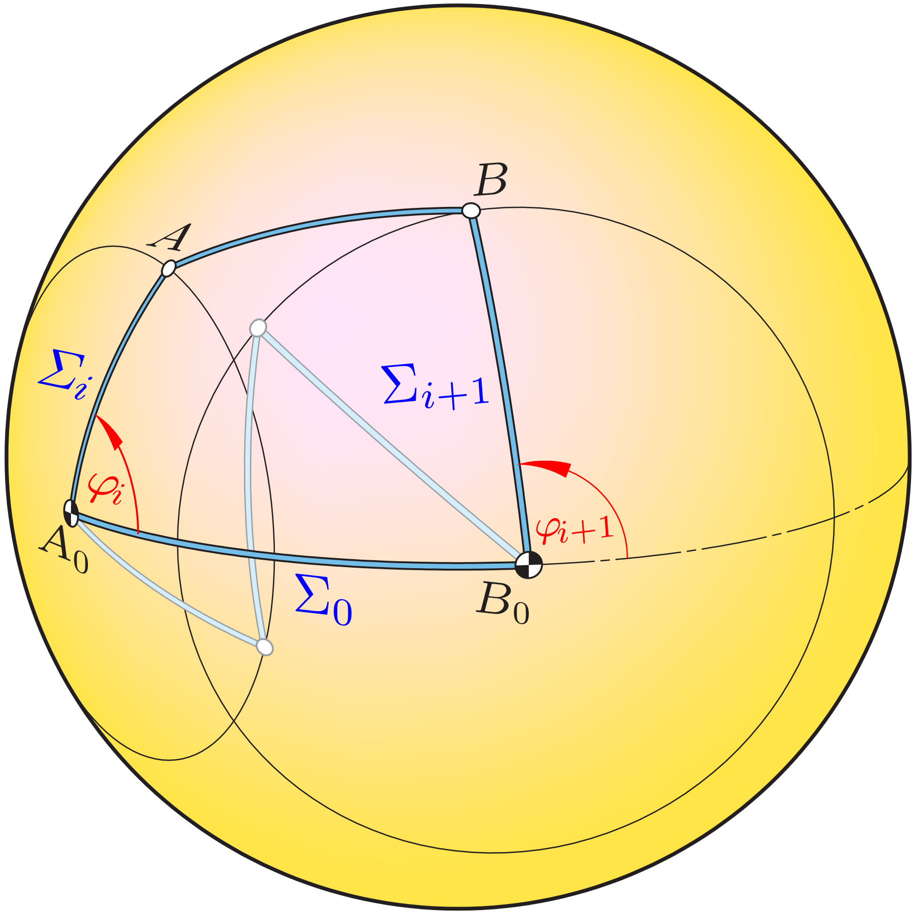
The polygons need not be planar

Kinematic interpretation:

The polygons represent different systems $\Sigma_0, \ldots, \Sigma_n$.
The sides a_i of $\boldsymbol{f}_0$ are instantaneous axes I_{i0} of the relative motions Σ_i / Σ_0 .

The relative motions Σ_{i+1}/Σ_i between consecutive systems are spherical four-bars mechanisms.

1. Definition of Kokotsakis meshes



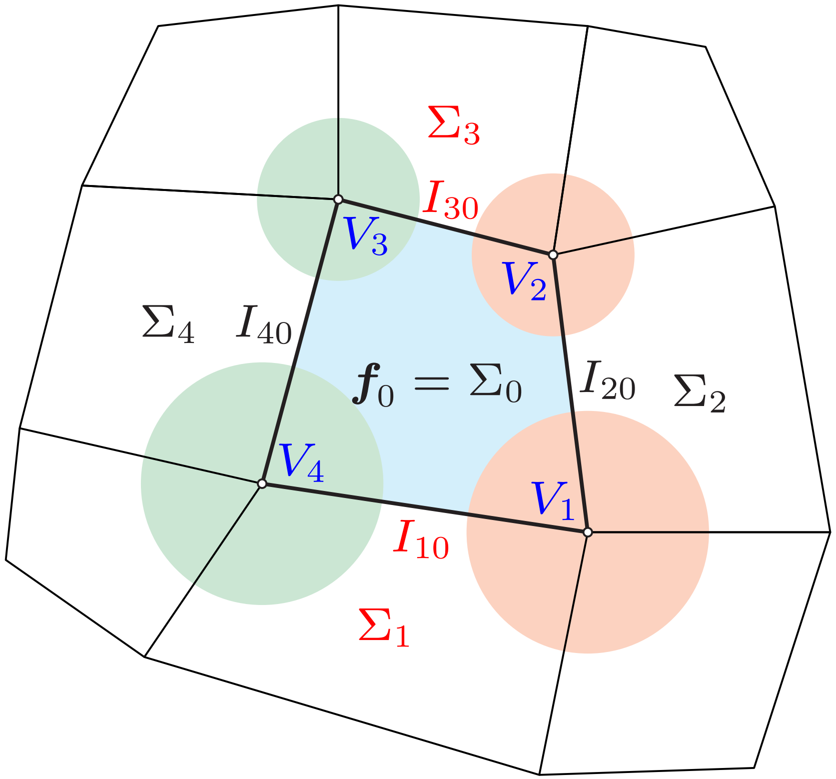
The transmission from Σ_i to the following Σ_{i+1}, $\varphi_i \mapsto \varphi_{i+1}$, is realized by a spherical four-bar:

To recall:

A spherical four-bar transmits the rotation about the center A_0 by the coupler AB non-uniformly to the rotation about B_0.

The two arms A_0A and B_0B represent consecutive systems Σ_i, Σ_{i+1}.

1. Definition of Kokotsakis meshes



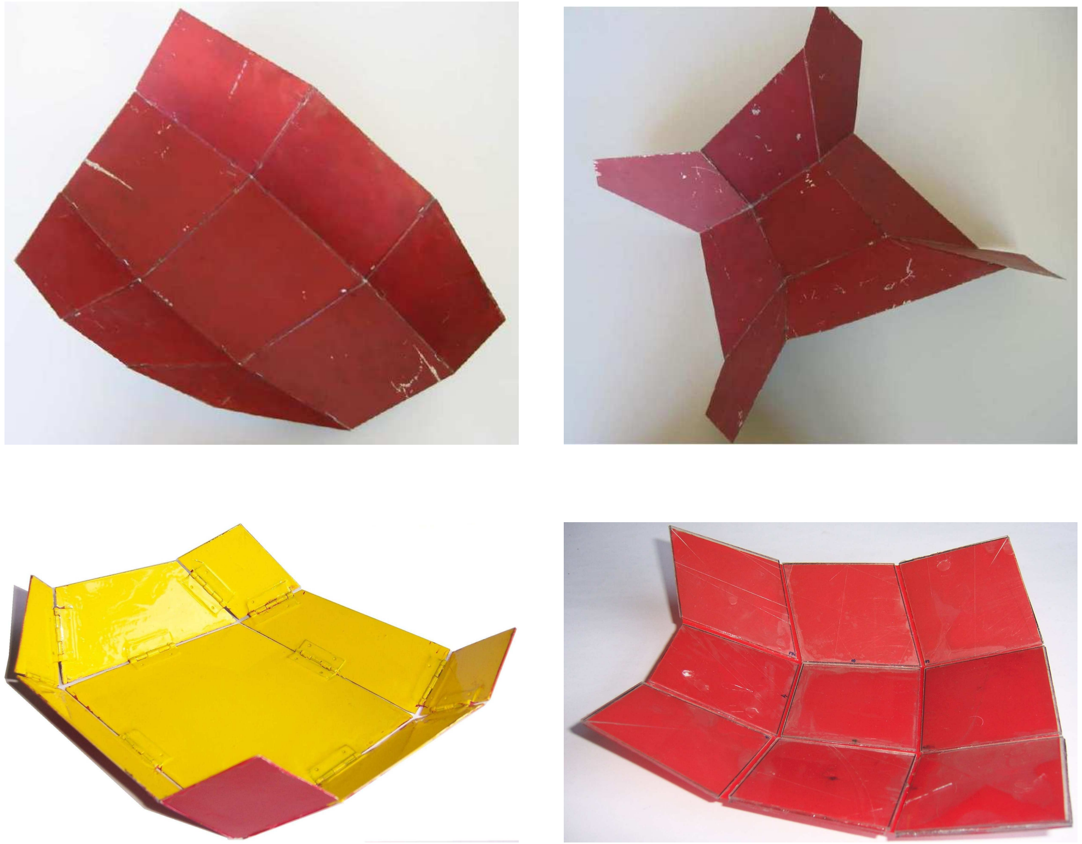
The edge lengths $\overline{V_1V_2}, \ldots, \overline{V_4V_1}$ of the central polygon $\boldsymbol{f}_0$ have no influence on the flexibility $\Longrightarrow$

Theorem: *A Kokotsakis-mesh for $n = 4$ is flexible if and only if the transmission $\Sigma_1 \mapsto \Sigma_3$ realized by the two four-bars (V_1, V_2) on the right hand side equals that via (V_3, V_4) on the left hand side.*

(we do not care about intersections between the involved quadrangles)

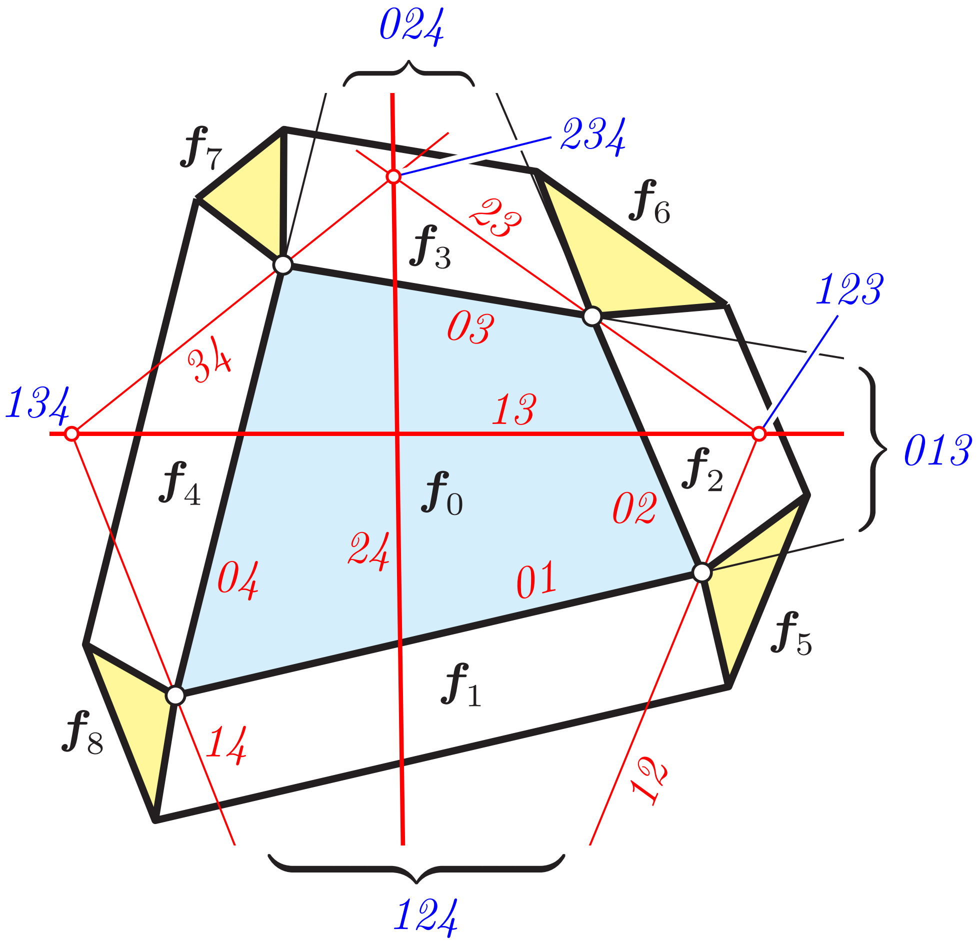
1. Definition of Kokotsakis meshes

Some models of flexible Kokotsakis meshes.



courtesy Nadja Posselt, Uwe Hanke, TU Dresden

1. Definition of Kokotsakis meshes



Theorem: (A. Kokotsakis (1932))

A Kokotsakis mesh is infinitesimally flexible $\iff$ the points of intersection between the traces of (f_1, f_3), (f_5, f_6) and (f_7, f_8) are collinear.

This is equivalent to the collinearity of the intersection points (f_2, f_4), (f_6, f_7) and (f_8, f_5).

The principle of "averaging" gives rise to snapping Kokotsakis meshes.

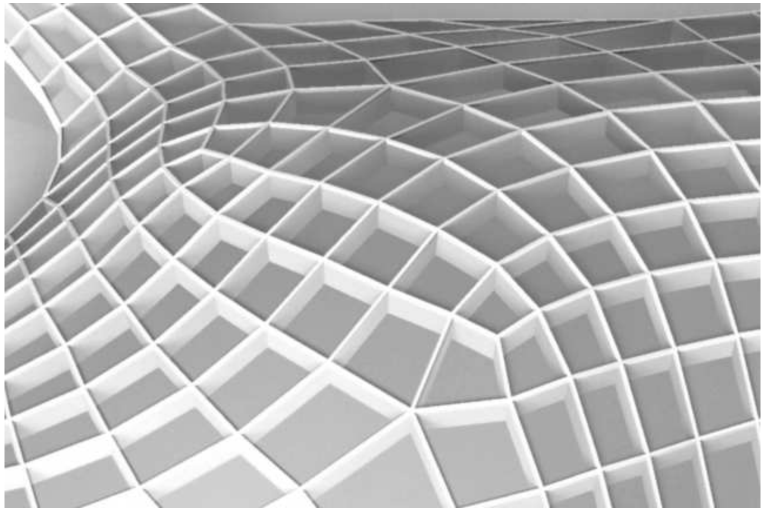
1. Definition of Kokotsakis meshes

In *discrete differential geometry* there is a interest on polyhedral structures composed of quadrilaterals (quadrilateral surfaces). If all quadrilaterals are planar, they form a *discrete conjugate net* = **quad mesh**.

Theorem: [Bobenko, Hoffmann, Schief 2008]

A discrete conjugate net in general position is continuously flexible $\iff$ *all its* 3×3 *complexes are continuously flexible.*

Bobenko et al., 2008:

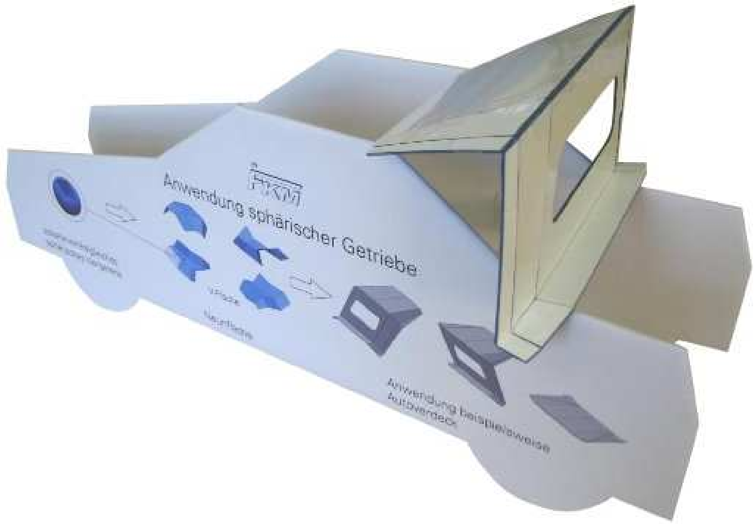


H. Pottmann, Y. Liu, J. Wallner, A. Bobenko, W. Wang: *Geometry of Multi-layer Freeform Structures for Architecture.* ACM Trans. Graphics **26** (3) (2007), SIGGRAPH 2007

"... the complete classification of flexible discrete conjugate nets *("quad meshes")* has not been achieved yet"

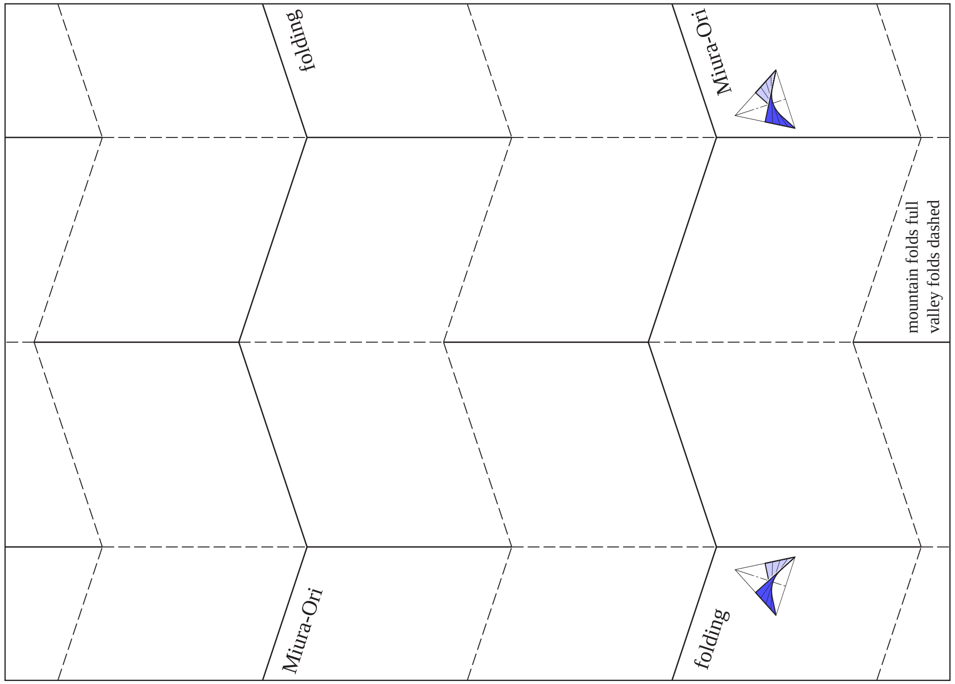
1. Definition of Kokotsakis meshes

Also the folding of the roof at cabrios is based on a flexible quad mesh



courtesy: Nadja Posselt
Diploma thesis, TU Dresden 2010

2. Three examples of flexible quad meshes



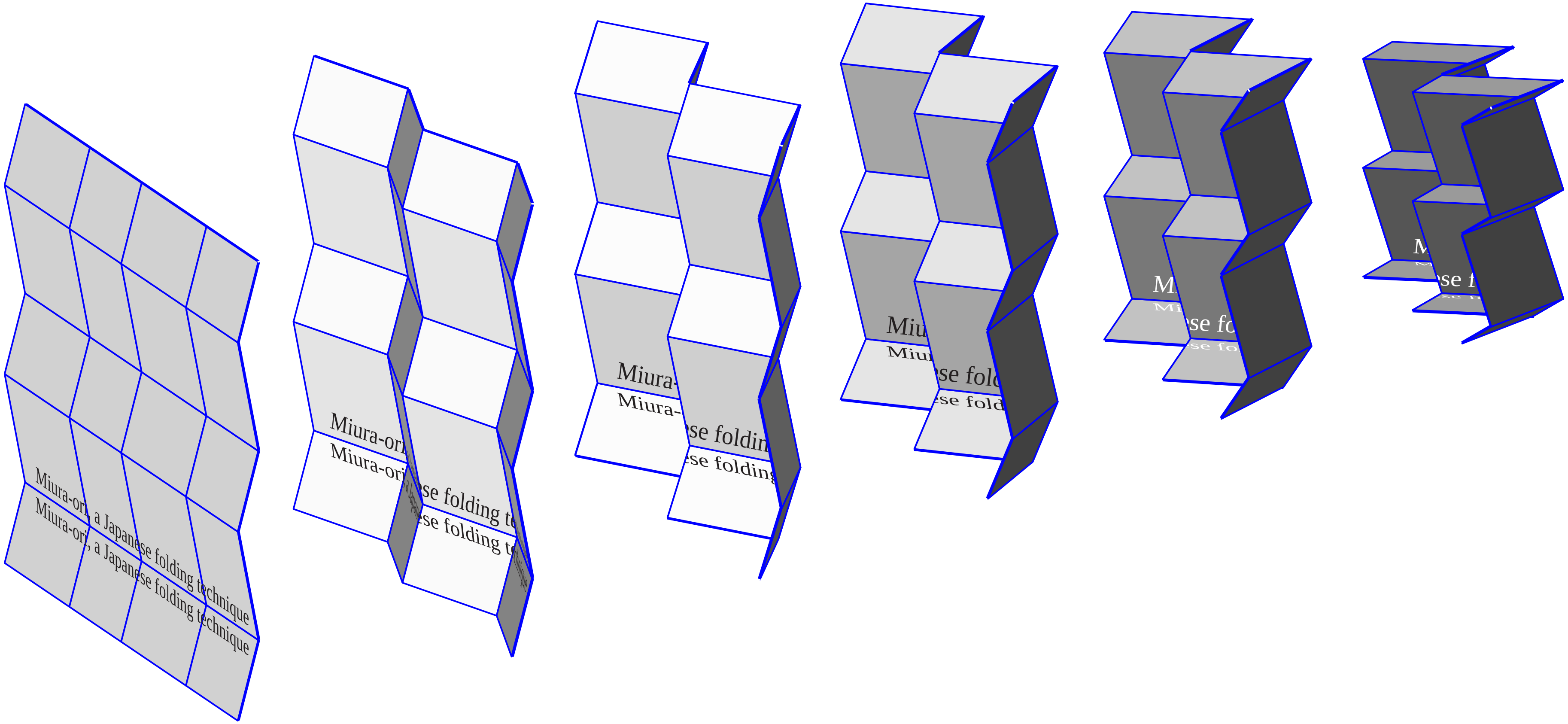
Unfolded miura-ori;
dashs are *valley folds*,
full lines are *mountain folds*

Miura-ori is a Japanese folding technique named after Prof. Koryo Miura, The University of Tokyo.

It is used for solar panels because it can be unfolded into its rectangular shape by pulling on one corner only.

On the other hand it is used as kernel to stiffen sandwich structures.

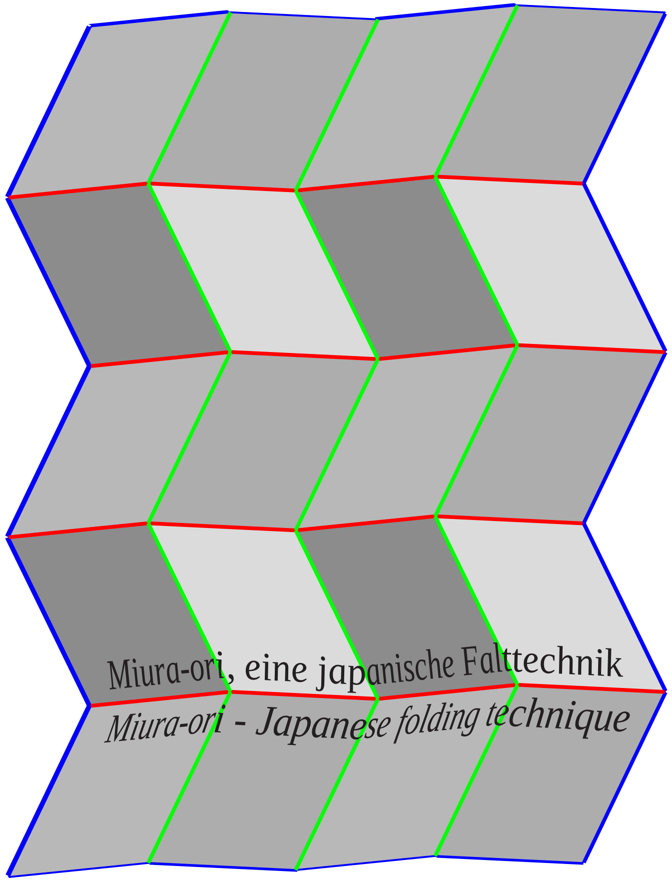
2. Three examples of flexible quad meshes



2. Three examples of flexible quad meshes

The edges of miura-ori constitute two sets of folds. The zig-zag lines placed in the horizontal planes $\varepsilon_1, \varepsilon_2$ are called horizontal folds. They are compounds of alternate valley and mountain folds.

The transversal folds are the vertical, are either pure valley folds or mountain folds. They are generated by iterated reflections in the horizontal planes $\varepsilon_1, \varepsilon_2$, hence located in vertical planes.



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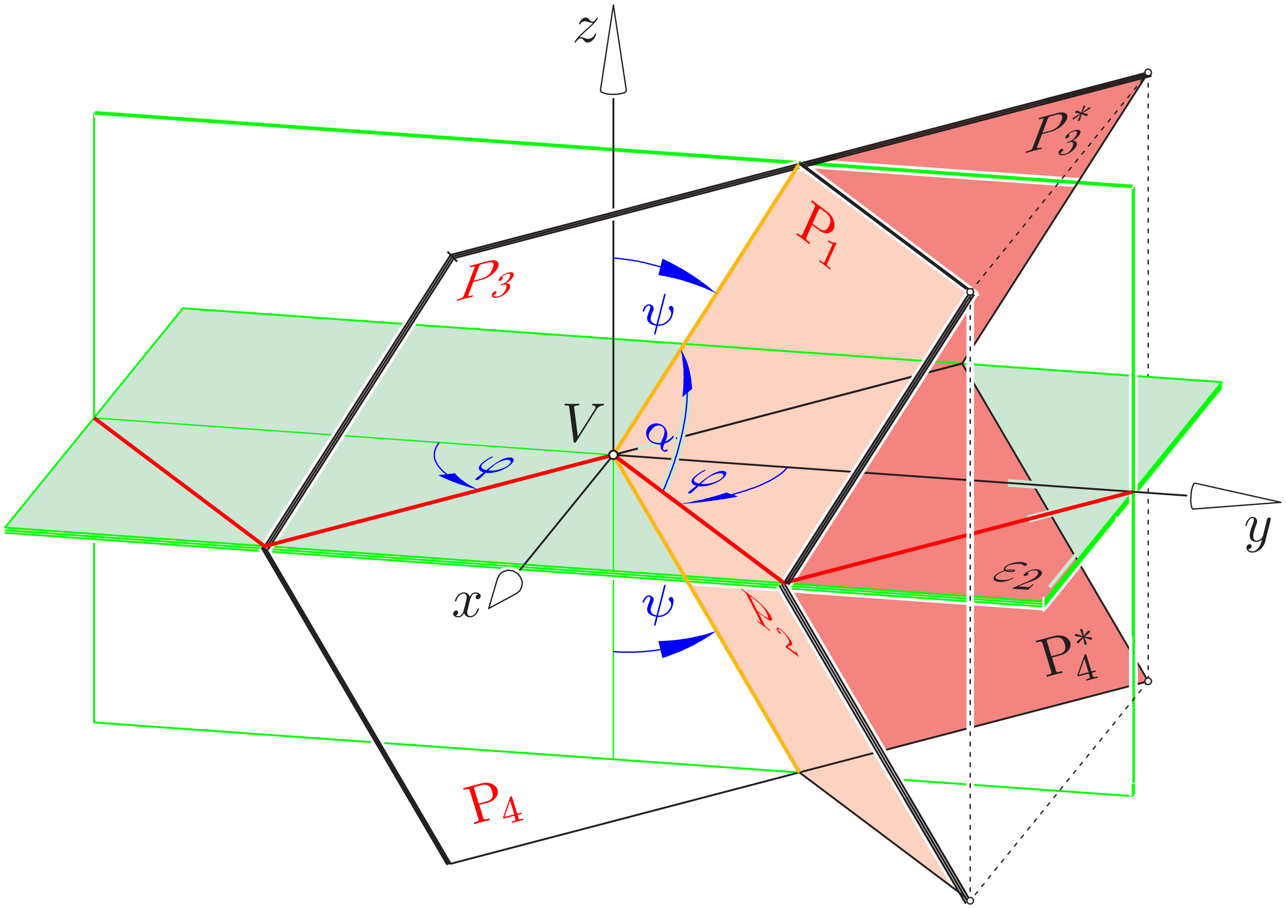
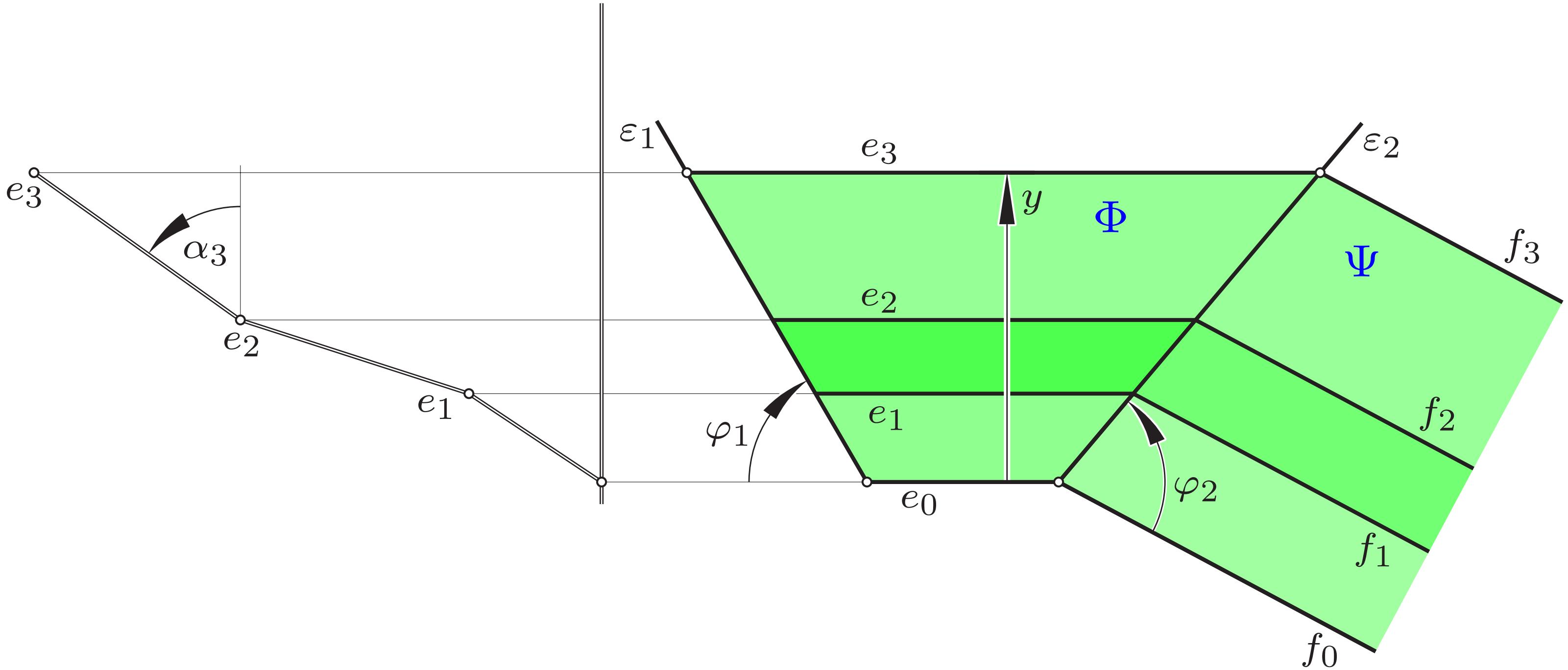


There is a hidden local symmetry at each vertex V:

The parallelograms P_1, P_2 with angle α and the elogations P_3^*, P_3^* of those with angle $180° - \alpha$ form a pyramid symmetric with respect to the fixed planes.

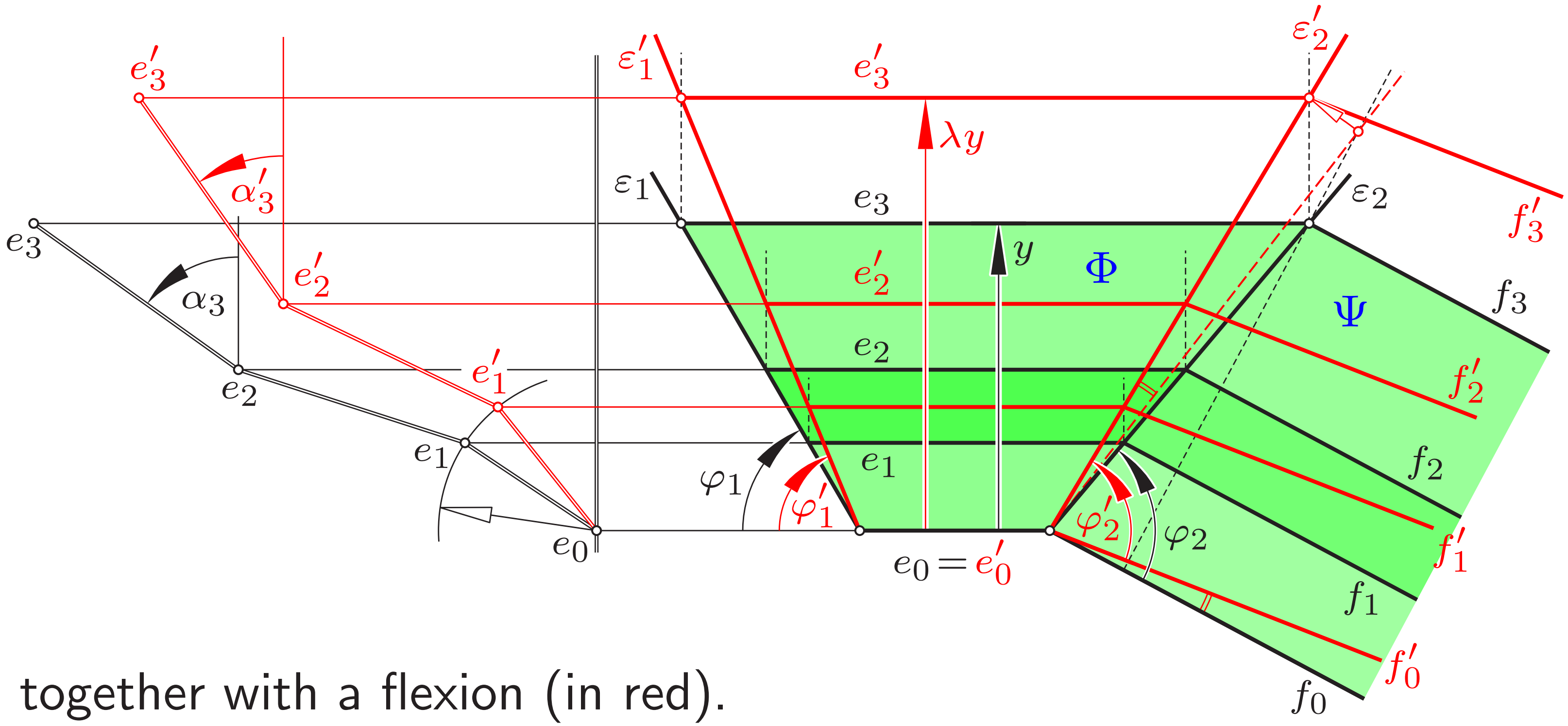
2. Three examples of flexible quad meshes



H. Graf, R. Sauer 1931: A **T-flat** is a compound of prisms $\Phi, \Psi, \ldots$ (see above: top view and side view. 'T' stands for 'trapezoid').

The horizontal folds $e_i, f_i, \ldots$ are located in horizontal planes, the vertical folds in vertical planes $\varepsilon_1, \varepsilon_2, \ldots$

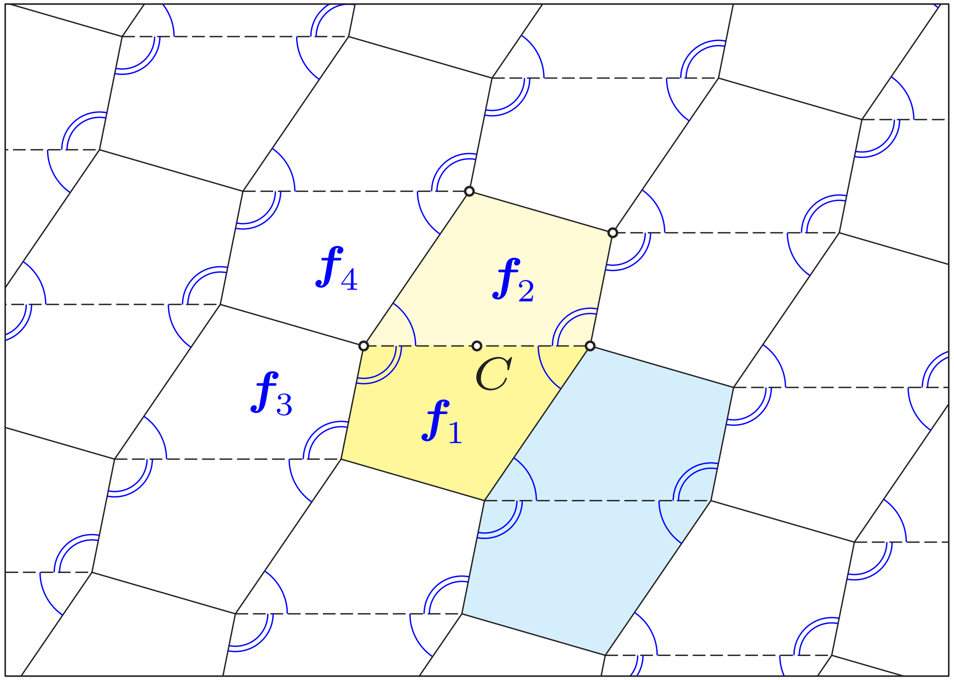
2. Three examples of flexible quad meshes



T-flat together with a flexion (in red).

The top view of Φ performs a scaling with factor λ orthogonal to e_0.
This implies analogous bendings of the other prisms $\Psi, \ldots$
$\Longrightarrow$ T-flats are continuously flexible.

2. Three examples of flexible quad meshes



A. KOKOTSAKIS, 1932
Athens

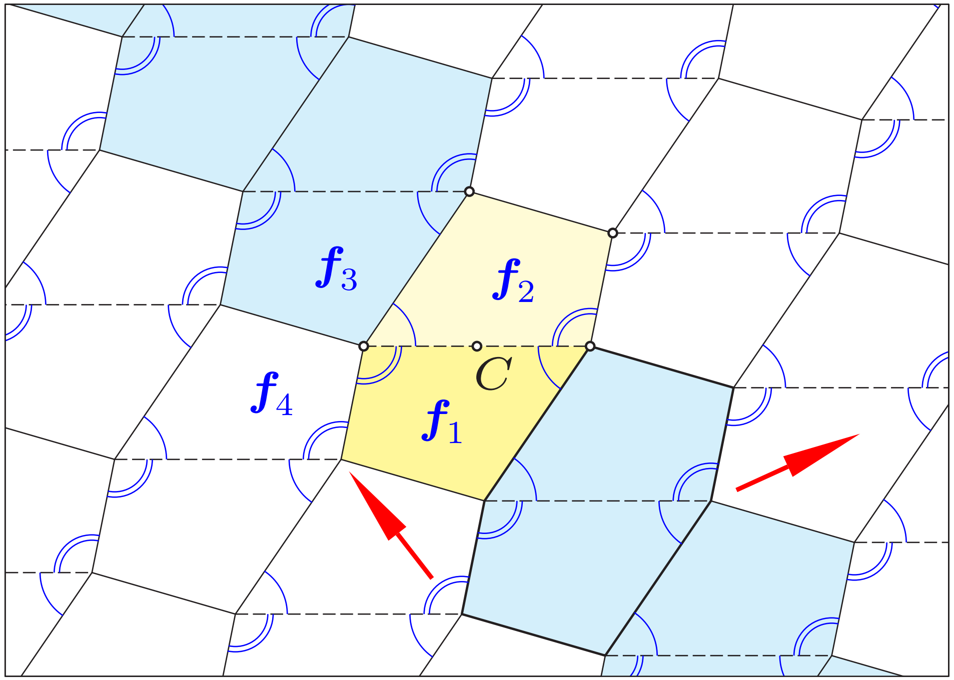
Any plane quadrangle is a tile for a regular tessellation of the plane.

It is obtained by applying

- iterated $180°$-rotations about the midpoints of the sides of an initial quadrangle or
- by applying iterated translations on a centrally symmetric hexagon.

For a convex f_1 this polyhedral structure is continuously flexible.

2. Three examples of flexible quad meshes



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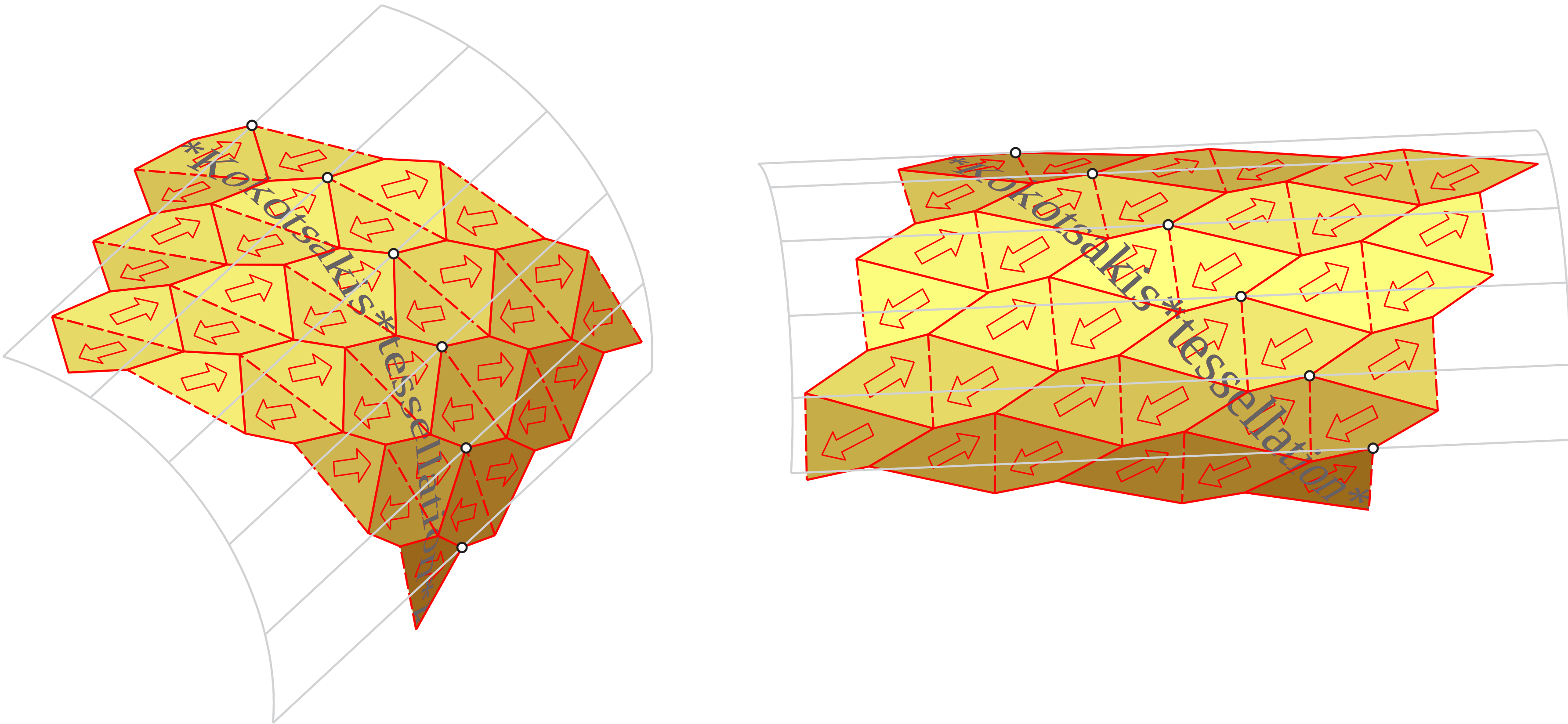
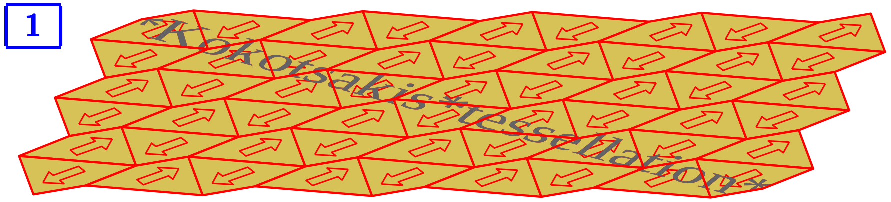


It can be proved that under continuous self-motions only such poses with vertices on right circular cylinders can be obtained (translations → helical motions).

2. Three examples of flexible quad meshes

1



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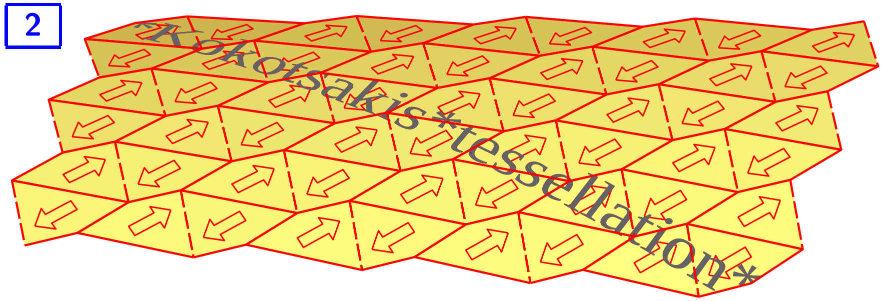

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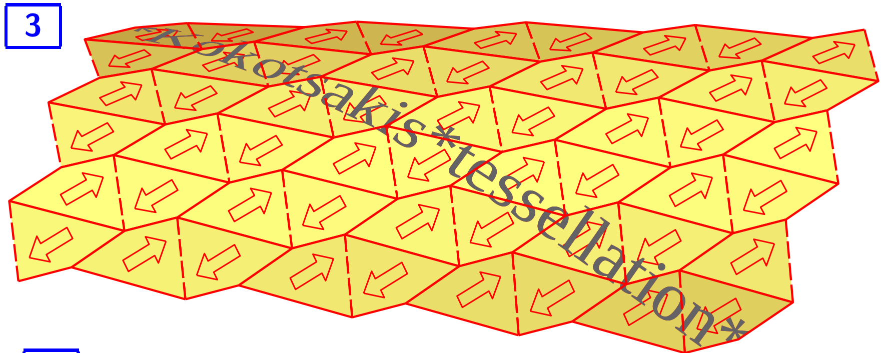

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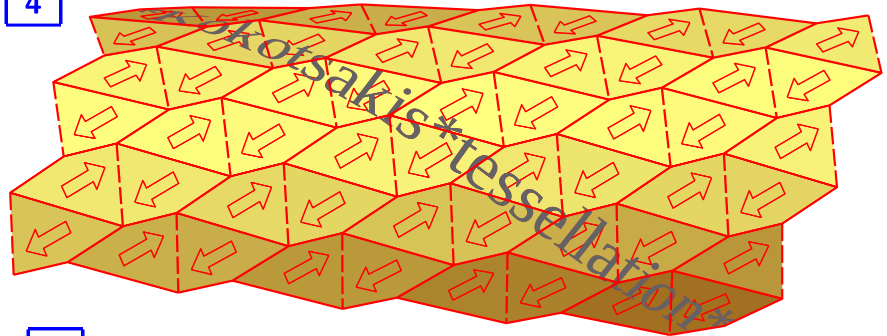

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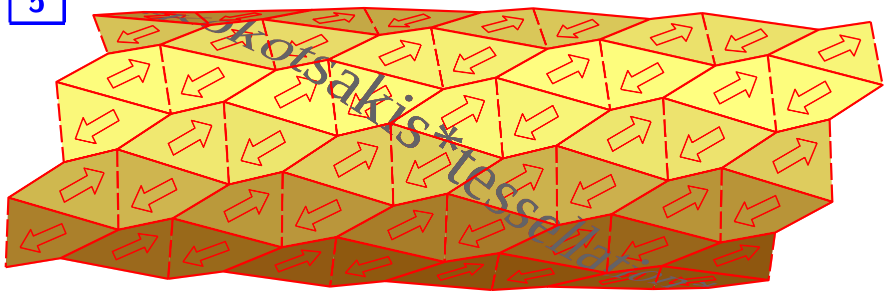

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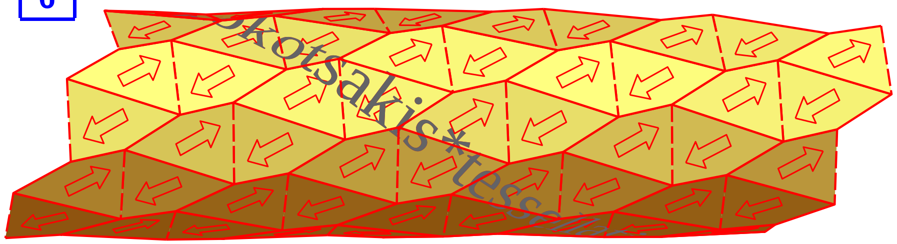

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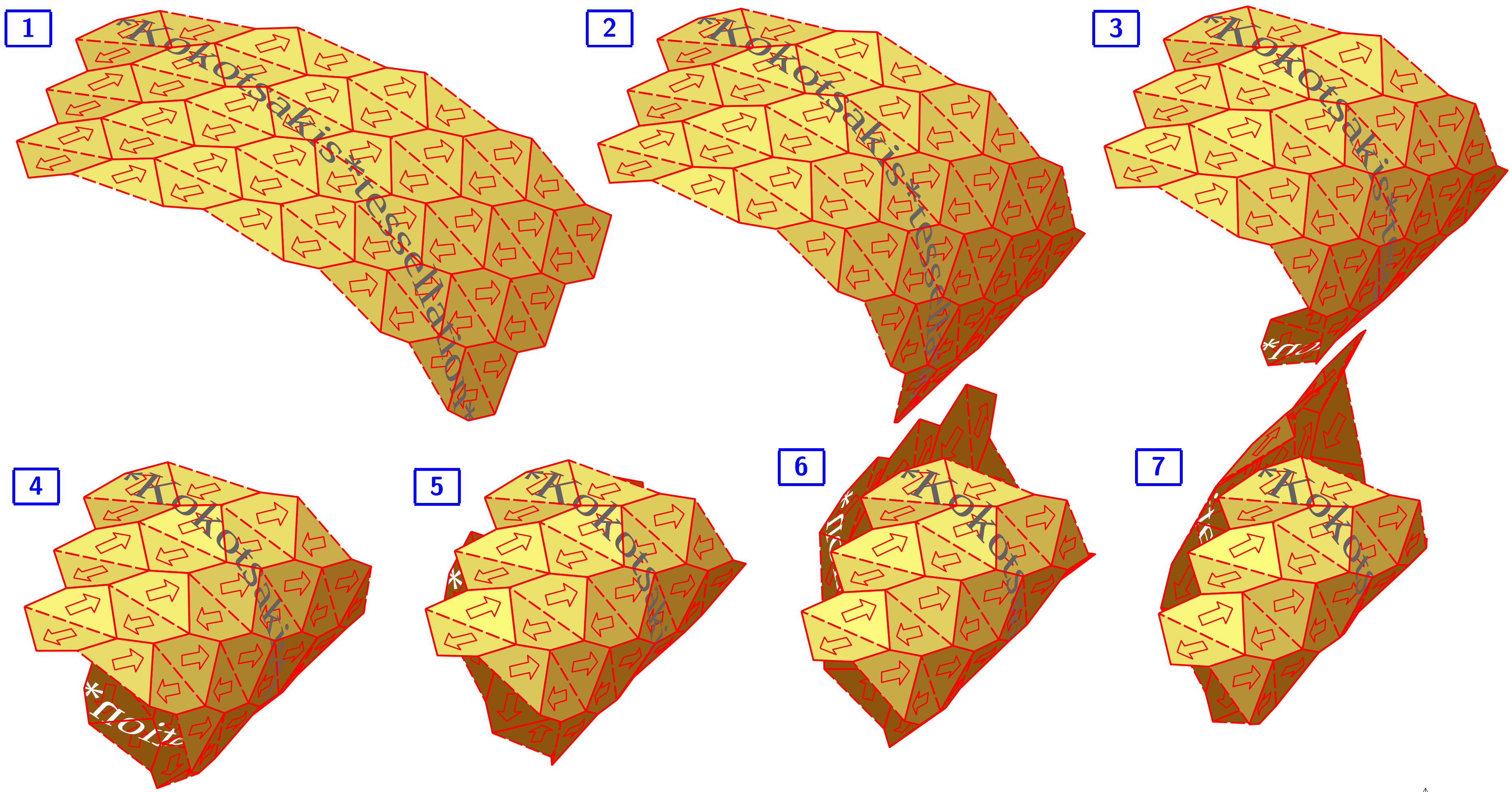
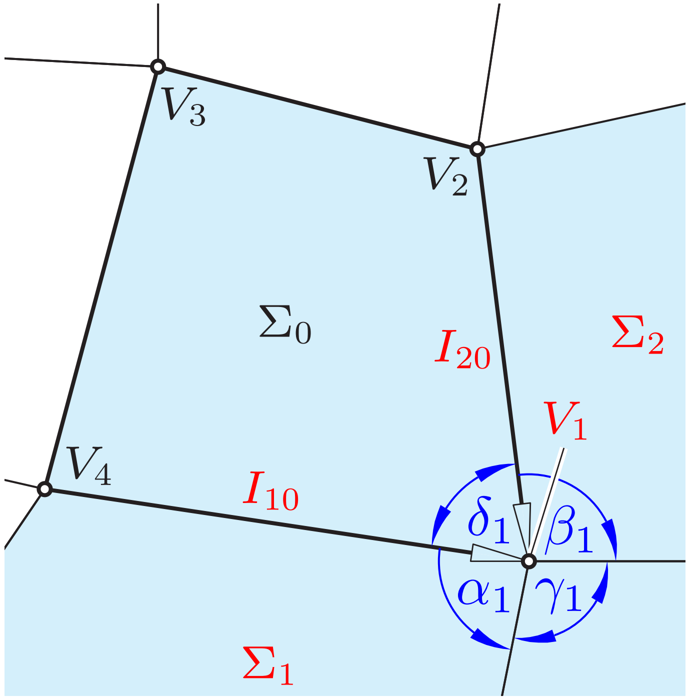
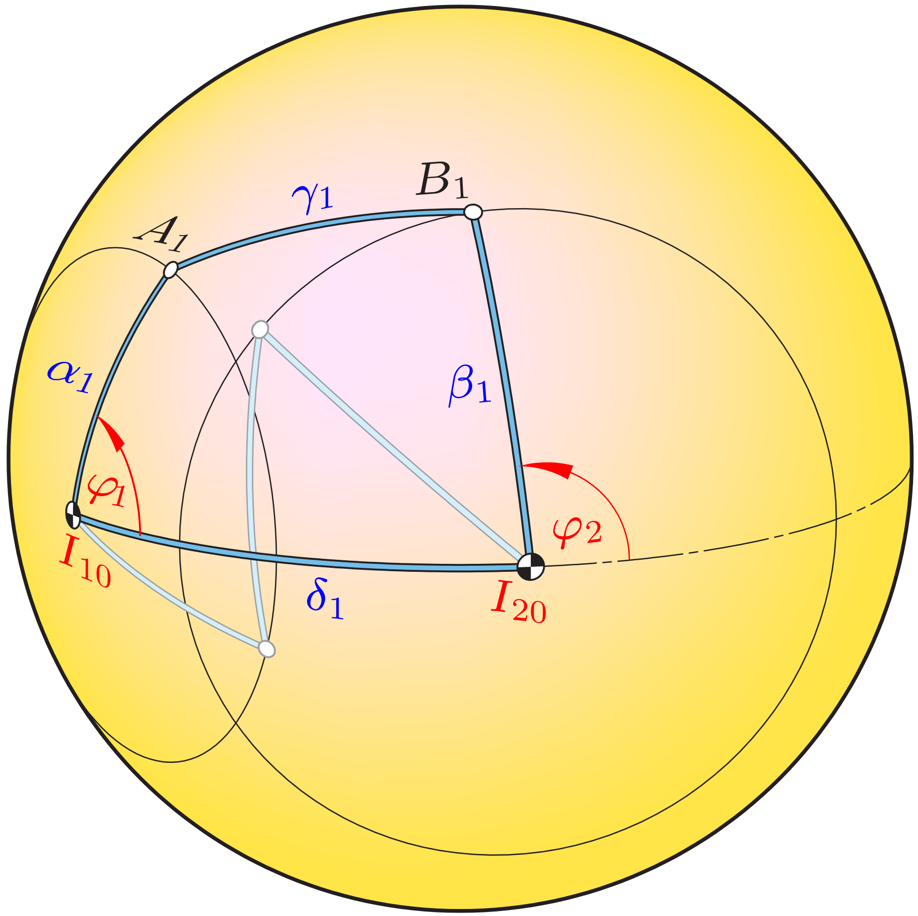


3. Transmission by one spherical four-bar

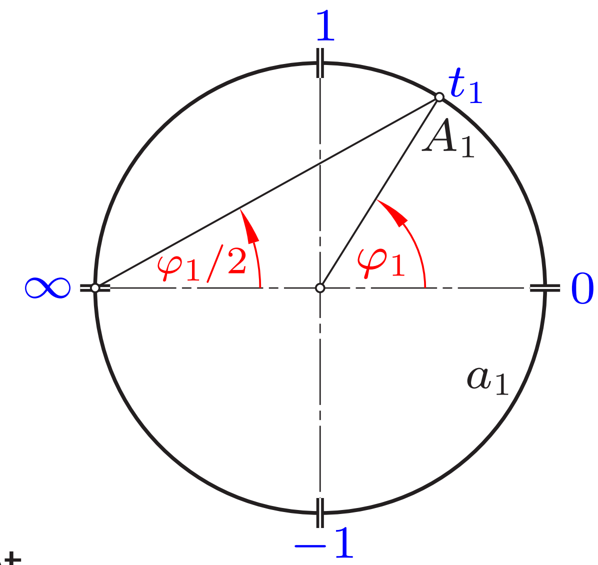


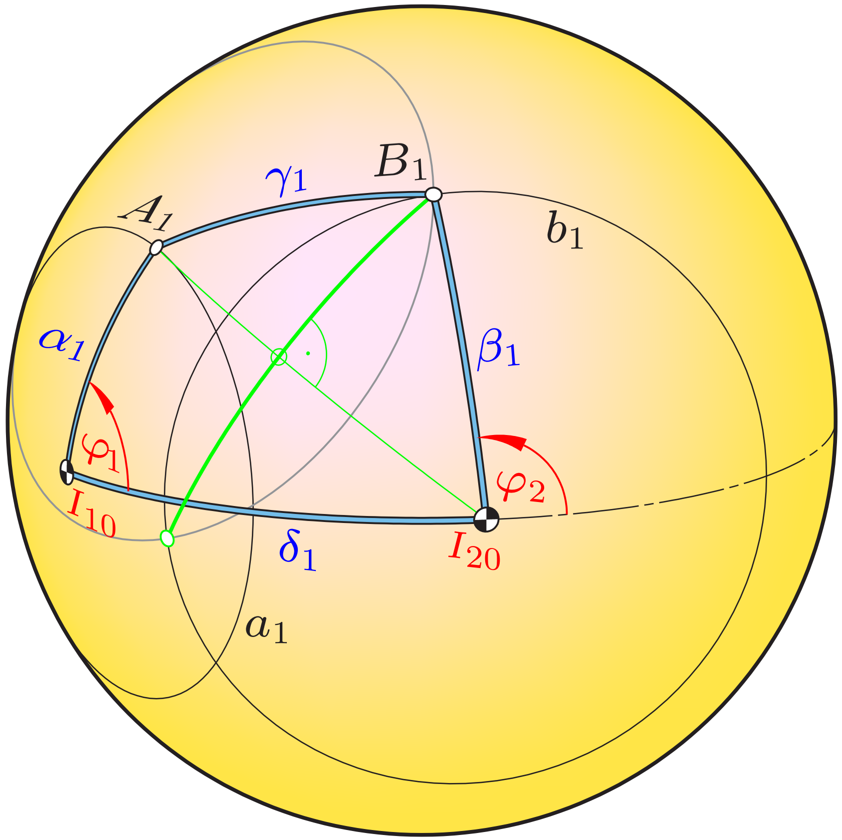


Four-bar motion Σ_2/Σ_1 and its spherical image

$$0 < \alpha_1, \beta_1, \gamma_1, \delta_1 < 180^\circ$$

3. Transmission by one spherical four-bar





We set

$$t_1 := \tan\frac{\varphi_1}{2}, \qquad t_2 := \tan\frac{\varphi_2}{2}\,.$$

t_1, t_2 are projective coordinates on the path circles a_1, b_1 of A_1 and B_1, resp., and obtain

$$c_{22}t_1^2t_2^2 + c_{20}t_1^2 + c_{02}t_2^2 + c_{11}t_1t_2 + c_{00} = 0 \quad \text{with} \quad c_{ik} = f(\alpha_1, \dots, \delta_1)$$

3. Transmission by one spherical four-bar

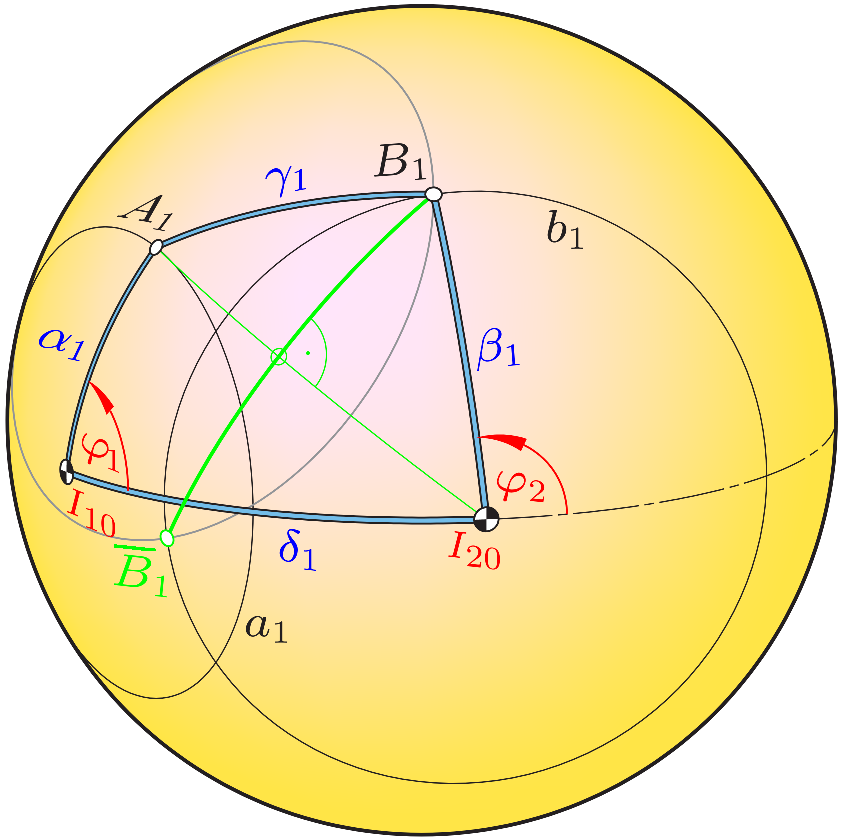
The transmission $\varphi_1 \mapsto \varphi_2$ by the four-bar linkage defines a 2-2-correspondence between the circles a_1 and b_1:

$$c_{22}t_1^2t_2^2 + c_{20}t_1^2 + c_{02}t_2^2 + c_{11}t_1t_2 + c_{00} = 0$$

. . . like in the plane.

$$(t_1 := \tan\frac{\varphi_1}{2}, \quad t_2 := \tan\frac{\varphi_2}{2})$$

On the sphere ambiguities arise as points can be replaced by their antipodes.



3. Transmission by one spherical four-bar

The coefficients in the biquadratic equation

$$c_{22}t_1^2t_2^2 + c_{20}t_1^2 + c_{02}t_2^2 + c_{11}t_1t_2 + c_{00} = 0$$

are:

$$c_{22} = \sin\tfrac{\alpha_1-\beta_1+\gamma_1+\delta_1}{2}\,\sin\tfrac{\alpha_1-\beta_1-\gamma_1+\delta_1}{2}$$

$$c_{20} = \sin\tfrac{\alpha_1+\beta_1+\gamma_1+\delta_1}{2}\,\sin\tfrac{\alpha_1+\beta_1-\gamma_1+\delta_1}{2}$$

$$c_{11} = -2\,\sin\alpha_1\,\sin\beta_1 \neq 0$$

$$c_{02} = \sin\tfrac{\alpha_1+\beta_1+\gamma_1-\delta_1}{2}\,\sin\tfrac{\alpha_1+\beta_1-\gamma_1-\delta_1}{2}$$

$$c_{00} = \sin\tfrac{\alpha_1-\beta_1+\gamma_1-\delta_1}{2}\,\sin\tfrac{\alpha_1-\beta_1-\gamma_1-\delta_1}{2}$$

3. Transmission by one spherical four-bar

The 2-2-correspondence

$$c_{22}t_1^2t_2^2 + c_{20}t_1^2 + c_{02}t_2^2 + c_{11}t_1t_2 + c_{00} = 0$$

depends only on the ratio of the coefficients.

Theorem:

For any spherical four-bar linkage the coefficients c_{ik} are algebraically dependent: c_{11} is a root of a 6th-degree polynomial with coefficients depending on $c_{00}, c_{02}, c_{20}, c_{22}$.

Conversely, in the complex extension any choice of coefficients in the biquadratic equation above defines the spherical four-bar linkage uniquely — up to replacement of vertices by their antipodes. However, the vertices need not be real.

3. Transmission by one spherical four-bar

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3. Transmission by one spherical four-bar

Particular cases of the 2-2-correspondence:

1) The 2-2-correspondence between a_1 and b_1 splits into two projectivities $\iff$ the quadrangle is a spherical isogram, i.e., $\beta_1 = \alpha_1$ and $\delta_1 = \gamma_1$ $(c_{00} = c_{22} = 0)$.

In this case (. . . **isogonal type**)

$$t \;\mapsto\; t_2 = \frac{\sin\alpha_1 \pm \sin\gamma_1}{\sin(\alpha_1 - \gamma_1)}\, t_1 \quad \text{for} \quad \alpha_1 \neq \gamma_1,\, \pi - \gamma_1$$

combines two linear functions.

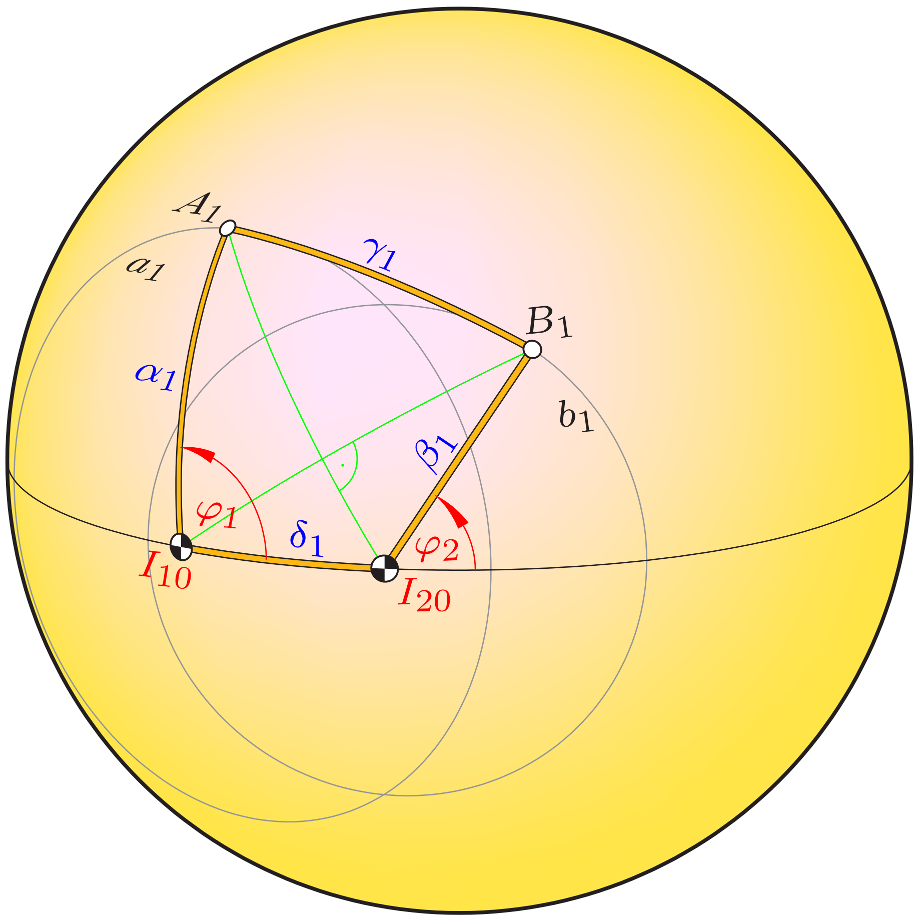
3. Transmission by one spherical four-bar

2) Under the condition

$$\cos\alpha_1\,\cos\beta_1 = \cos\gamma_1\,\cos\delta_1$$

(equivalent to $\det(c_{ik}) = 0$) each quadrangle has orthogonal diagonals (... **orthogonal type**).

The 2-2-correspondence maps pairs of points on a_1 aligned with I_{20} onto pairs of points on b_2 located on the orthogonal line through I_{10} .



3. Transmission by one spherical four-bar

3) Deltoid type:

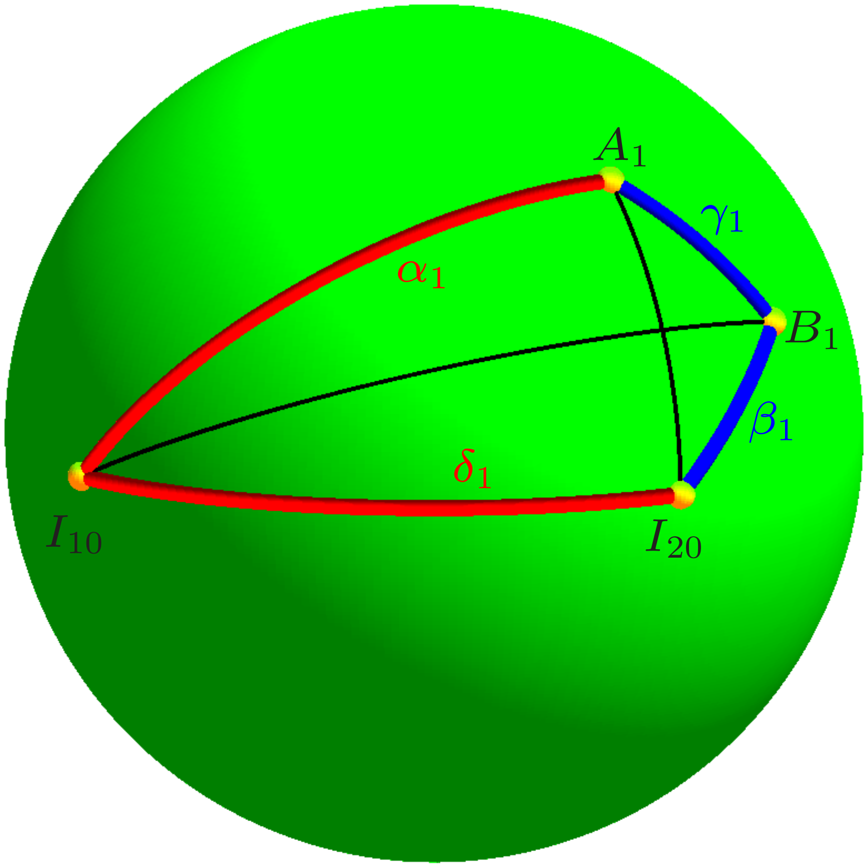
$$\alpha_1 = \delta_1 \implies c_{00} = c_{02} = 0\,.$$

The 2-2-correspondence splits:

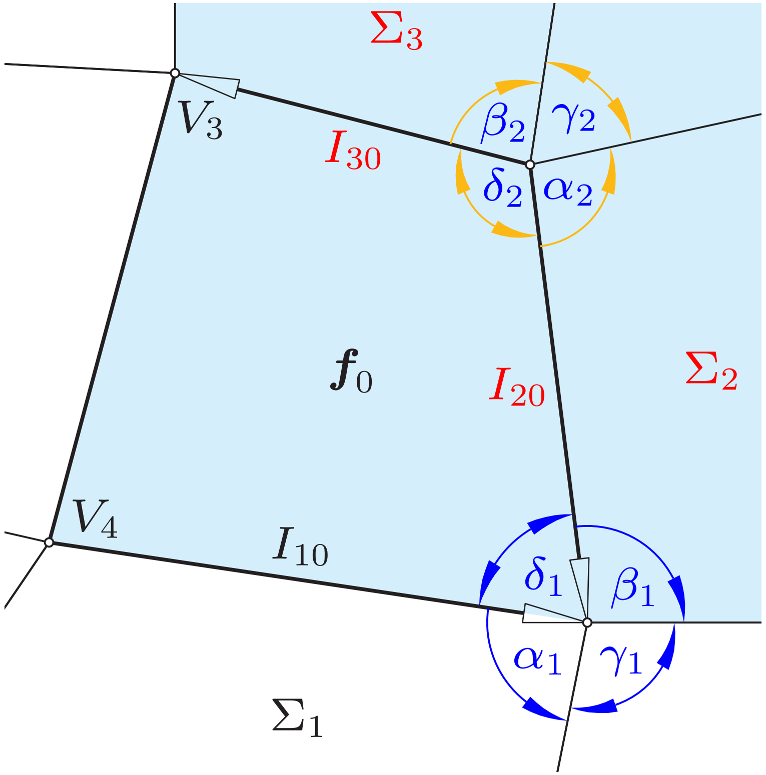
$$t_1\left(c_{22}t_1t_2^2 + c_{20}t_1 + c_{11}t_2\right) = 0;$$

a) $t_1 = 0$ corresponds to all $t_2 \in \mathbb{R}$,

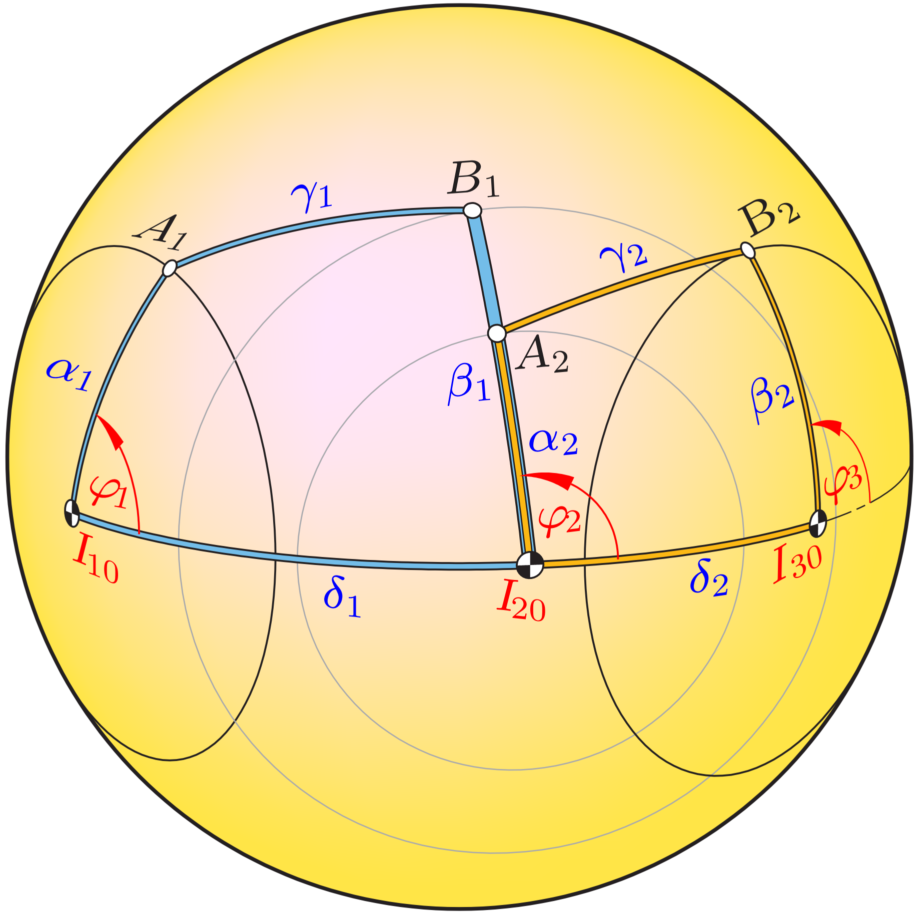
b) 1-2-correspondence.



4. Composition of spherical four-bars



Composition of two four-bars Σ_2/Σ_1 and Σ_3/Σ_1 and their spherical images



4. Composition of spherical four-bars

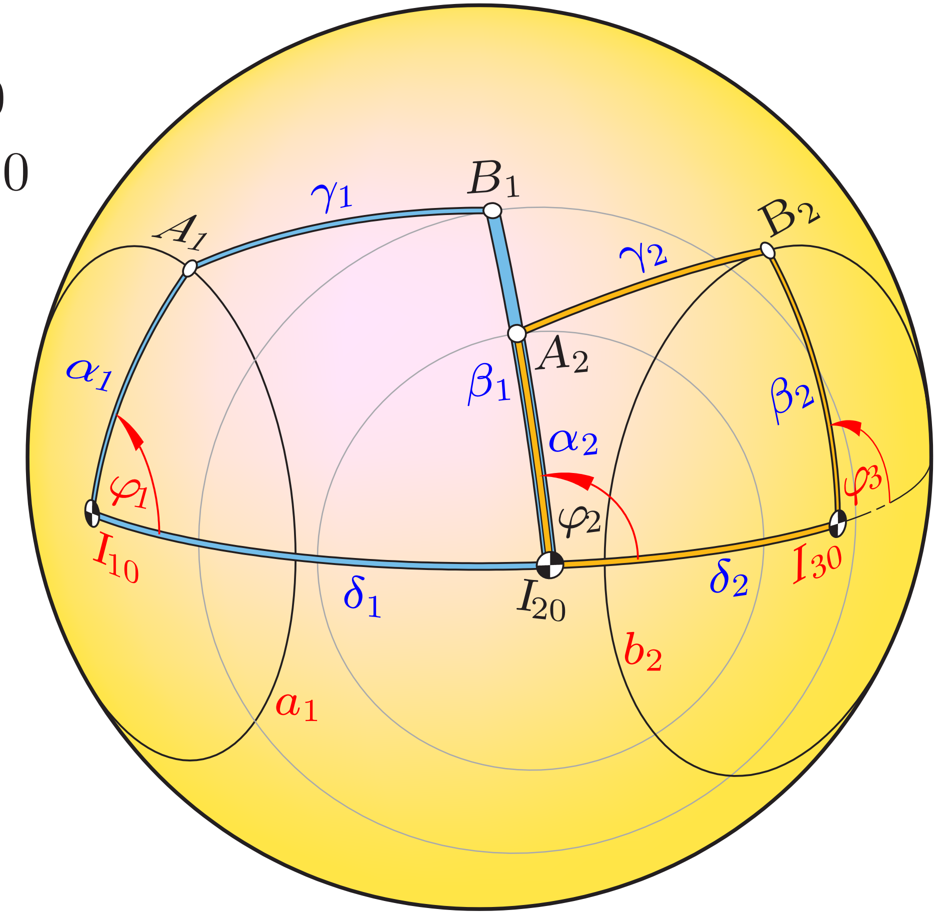
$$c_{22}t_1^2t_2^2 + c_{20}t_1^2 + c_{02}t_2^2 + c_{11}t_1t_2 + c_{00} = 0$$
$$d_{22}t_3^2t_2^2 + d_{20}t_3^2 + d_{02}t_2^2 + d_{11}t_3t_2 + d_{00} = 0$$

The four-bar transmissions are equivalent to these two bilinear equations.

We eliminate t_2 by computing the resultant with respect to t_2. Thus we obtain a biquartic equation in

$$t_1 = \tan\frac{\varphi_1}{2} \text{ and } t_3 = \tan\frac{\varphi_3}{2},$$

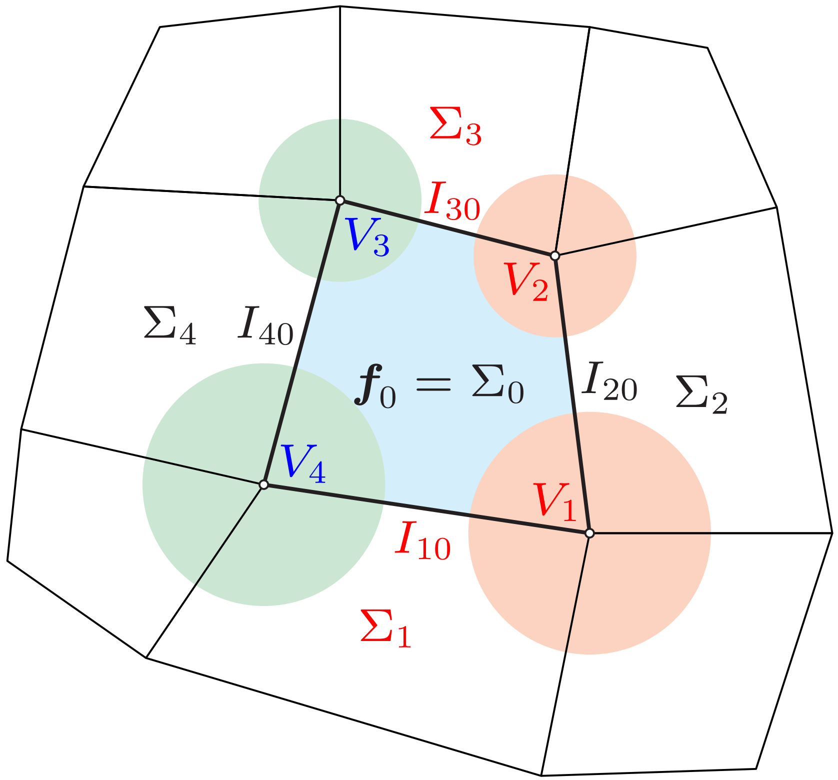
i.e., a **4-4-correspondence** between $A_1 \in a_1$ and $B_2 \in b_2$.



4. Composition of spherical four-bars

Continuous flexibility of a Kokotsakis mesh for $n = 4$ means:

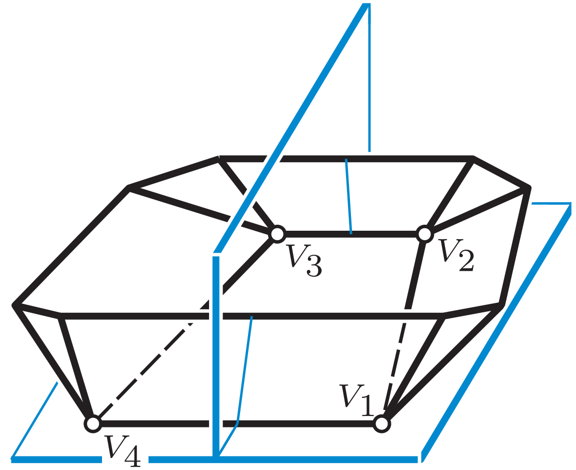
The 4-4-correspondence or – in the reducible case – one of its components can be decomposed in two different ways.



4. Composition of spherical four-bars

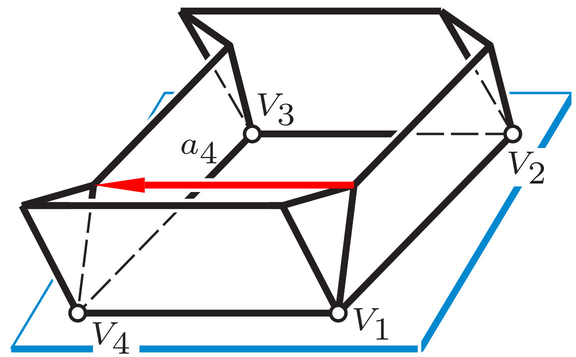
I. Planar-symmetric type (Kokotsakis 1932):

The reflection in the plane of symmetry of V_1 and V_4 maps each horizontal fold onto itself while the two vertical folds are exchanged.



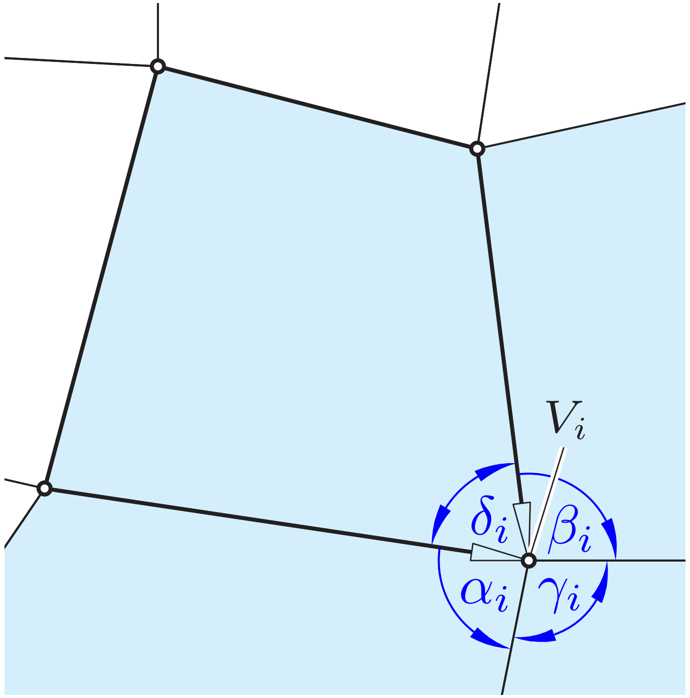
II. Translational type:

There is a translation $V_1 \mapsto V_4$ and $V_2 \mapsto V_3$ mapping the three faces on the right hand side onto the triple on the left hand side.



The composition of two linear functions $t_1 \mapsto t_2$ and $t_2 \mapsto t_3$ is again linear $\Longrightarrow$

4. Composition of spherical four-bars



III: Isogonal type: $(n \geq 4)$

Miura-ori is a special case of

Theorem: [Kokotsakis 1932]
A Kokotsakis mesh is flexible when at each vertex V_i opposite angles are either equal or complementary, i.e.,

$$\alpha_i = \beta_i, \quad \gamma_i = \delta_i \text{ or}$$
$$\alpha_i = \pi - \beta_i, \quad \gamma_i = \pi - \delta_i \,.$$

A quad mesh where all vertices are of this type is continuously flexible and called Voss surface (Kokotsakis, Graf, Sauer)

4. Composition of spherical four-bars

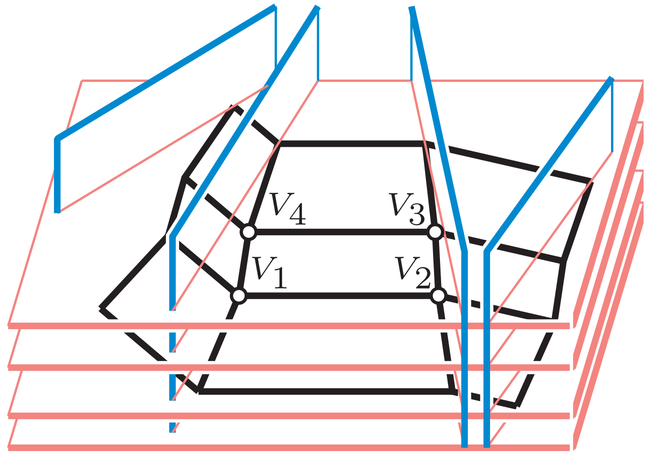
IIIa. Generalized isogonal type:

(A. Kokotsakis (1932): At each vertex opposite angles are congruent or complementary).

G. Nawratil (2010): At at least two of the four pyramides opposite angles are congruent.

IV. Orthogonal type (Graf, Sauer 1931):

Here the horizontal folds are located in parallel (say: horizontal) planes, the vertical folds in vertical planes (T-flat).

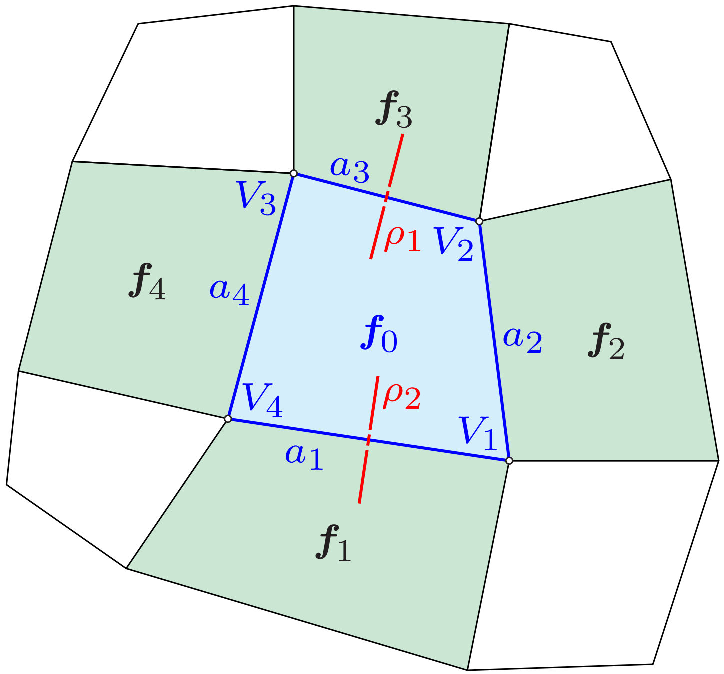


4. Composition of spherical four-bars

V. Line-symmetric type (H.S. 2009):

A line-reflection maps the pyramide at V_1 onto that of V_4; another one exchanges the pyramides at V_2 and V_3.

This includes Kokotsakis' example of a flexible tessellation.



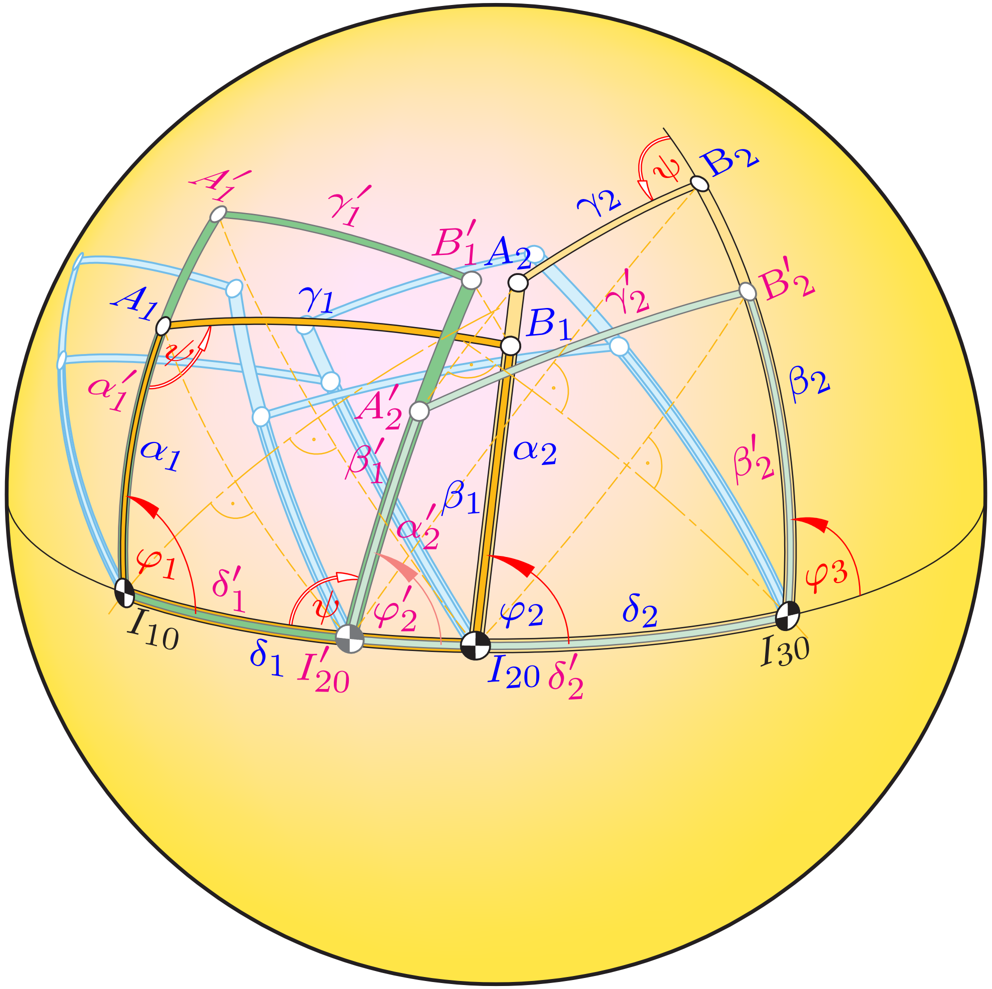
4. Composition of spherical four-bars

Under the conditions (case V)

$\alpha_1 + \beta_2 = \delta_1 + \delta_2$

$\mathrm{s}\alpha_1\mathrm{s}\gamma_1 : \mathrm{s}\beta_2\mathrm{s}\gamma_2 = \mathrm{s}\beta_1\mathrm{s}\delta_1 : \mathrm{s}\alpha_2\mathrm{s}\delta_2 =$
$(\mathrm{c}\beta_1\mathrm{c}\delta_1 - \mathrm{c}\alpha_1\mathrm{c}\gamma_1) : (\mathrm{c}\beta_2\mathrm{c}\gamma_2 - \mathrm{c}\alpha_2\mathrm{c}\delta_2)$

the 4-4-correspondance between t_1 and t_3 can be decomposed in two ways in the product of two 2-2-correspondences.



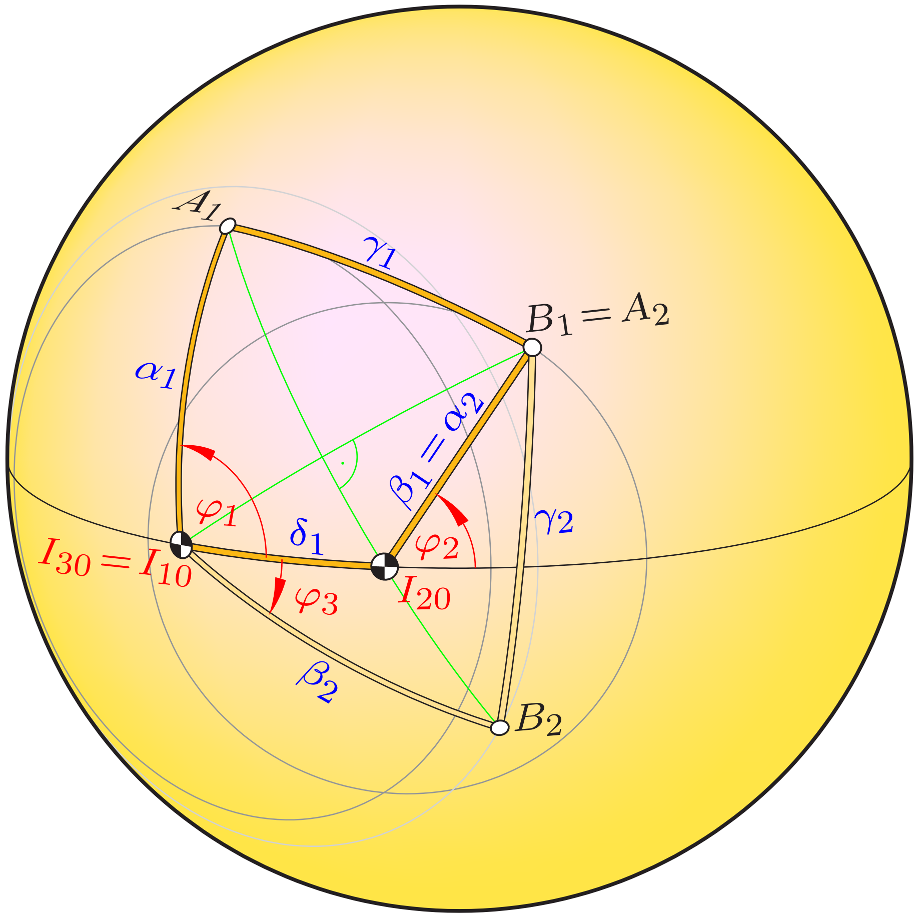
4. Composition of spherical four-bars

Under the conditions (case IV)

$$\cos\alpha_1 \ \cos\beta_1 = \cos\gamma_1 \ \cos\delta_1\,, \quad \alpha_2 = \beta_1\,,$$
$$\cos\alpha_2 \ \cos\beta_2 = \cos\gamma_2 \ \cos\delta_2\,, \quad \delta_2 = -\delta_1\,,$$

both four-bars share the orthogonal diagonals.

Due to GRAF and SAUER (1931) there is a second decomposition of the same kind; all four-bars share one diagonal (spherical DIXON mechanism).



4. Composition of spherical four-bars

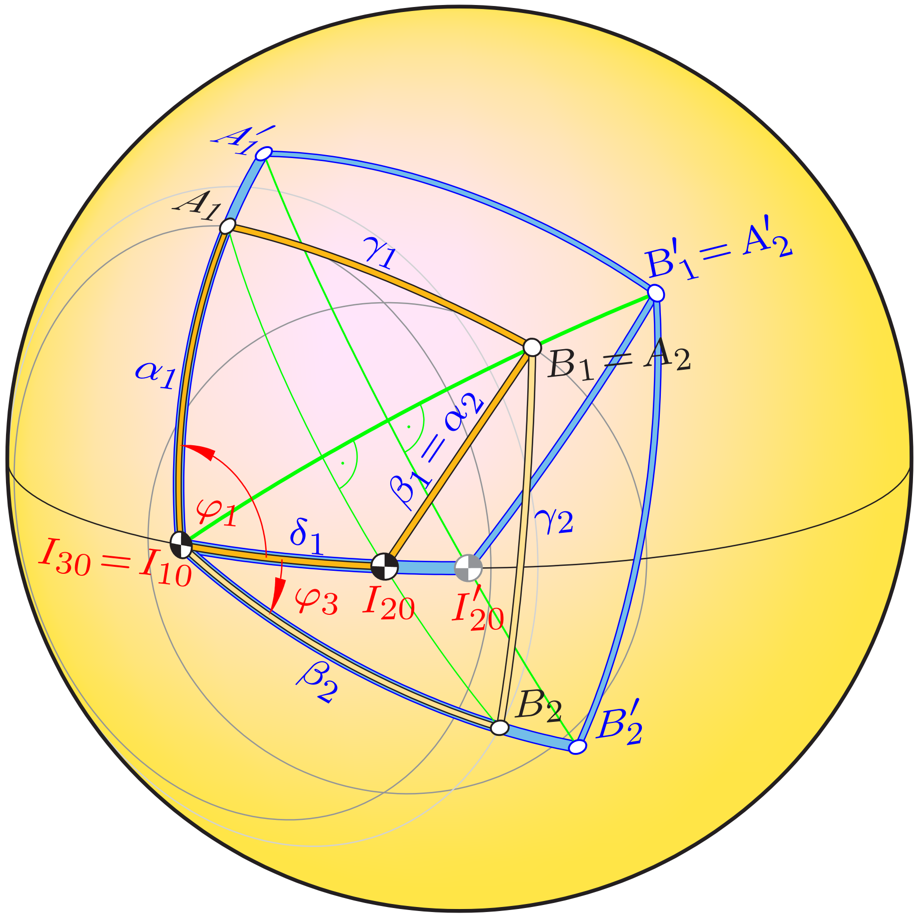
The 4-4-correspondence is the square of a 2-2-correspondence

$$c_{21}t_1^2t_3 + c_{12}t_1t_3^2 + c_{10}t_1 + c_{01}t_3 = 0$$

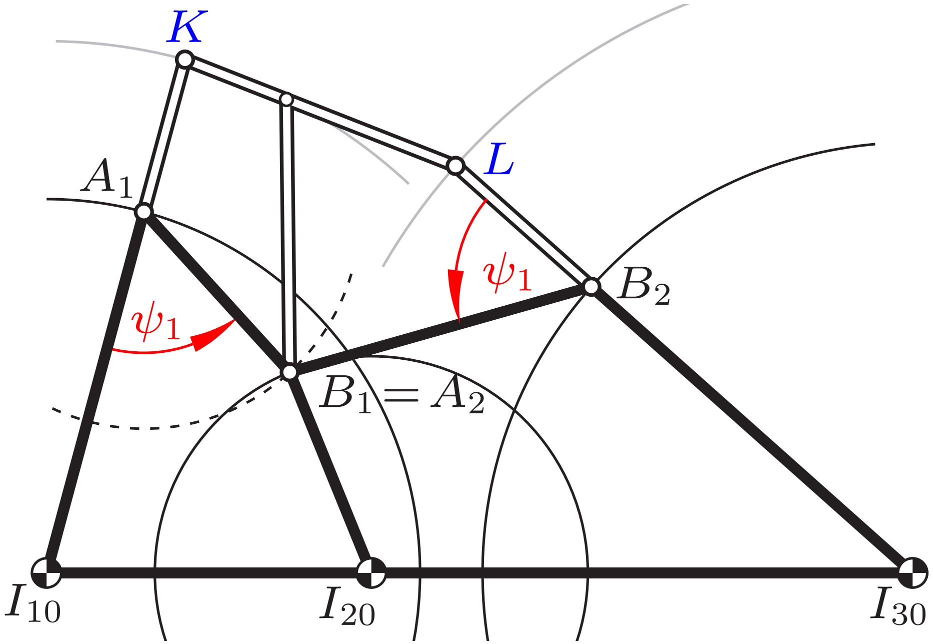
with coefficients depending only on $\tan\alpha_1$, $\tan\delta_1$, $\tan\beta_2$.

In all known non-trivial examples (III, IV, V) the 4-4-correspondence between t_1 and t_3 is **reducible**.

There is a new example of a reducible composition:

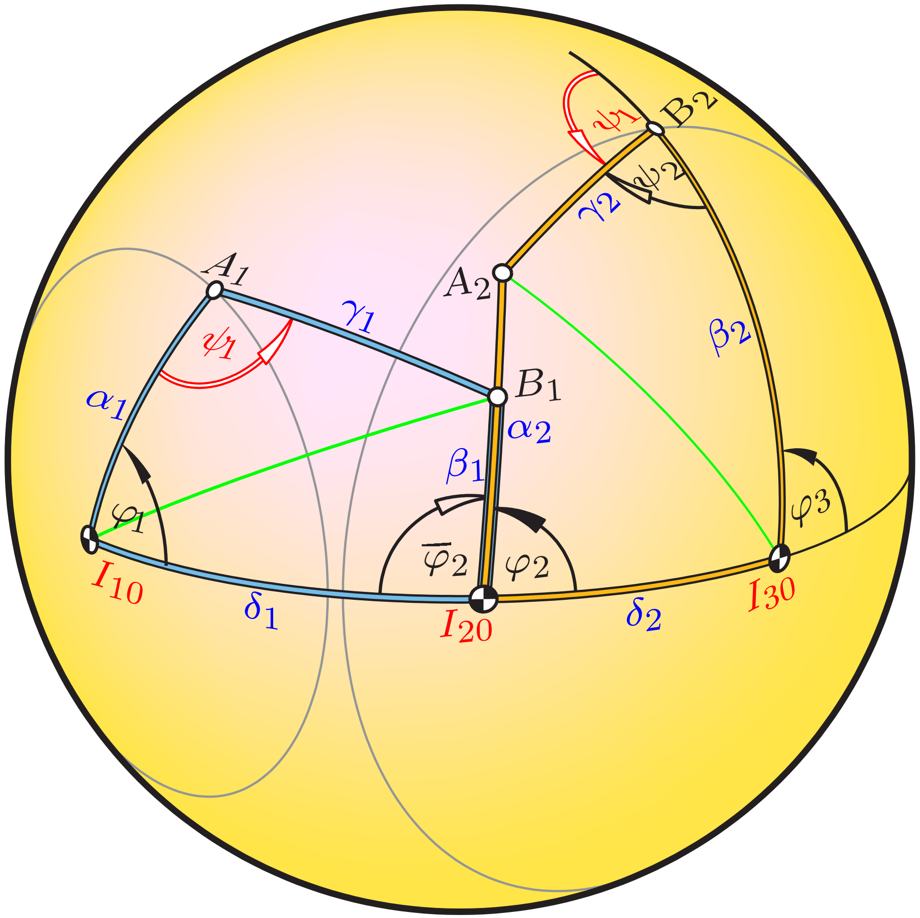


5. Flexibility vs. reducibility of meshes



BURMESTER's focal mechanism

Right hand figure: Reducible spherical composition obeying DIXON's angle condition for ψ_1



5. Flexibility vs. reducibility of meshes

For the composition of two spherical four-bars Dixon's angle condition $\sphericalangle\, I_{10}A_1B_1 = \pm \sphericalangle\, \overline{I}_{30}B_2A_2$ is equivalent to the statement that the discriminants of both 2-2-correspondences with respect to t_2

$$D_1 = (c_{11}t_2)^2 - 4(c_{22}t_2^2 + c_{20})(c_{02}t_2^2 + c_{00}) \quad \text{and}$$
$$D_2 = (d_{11}t_2)^2 - 4(d_{22}t_2^2 + d_{02})(d_{20}t_2^2 + d_{00})$$

are proportional.

Then the **4-4-correspondence is reducible**.

5. Flexibility vs. reducibility of meshes

Theorem: (G. NAWRATIL, 2011)

There are 4 non-trivial cases where the 4-4-correspondence is reducible:

1. Isogonal case: *One of the spherical quadrangles is isogonal.*

2. Dixon case: *The two spherical four-bars obey* DIXON*'s angle condition.*

3. Orthogonal case: *Both spherical quadrangles are orthogonal and share one diagonal (T-type).*

4: Deltoid case: *One of the quadrangles is a deltoid.*

5. Flexibility vs. reducibility of meshes

Conjecture:

Apart from the trivial translatory type I and planar-symmetrical type II there is no continuously flexible Kokotsakis-mesh with irreducible 4-4-correspondence.

Pro-arguments: The (complete) 4-4-correspondence (most probably) defines the 10 coefficients $c_{00}, \ldots, d_{22}$ of its components uniquely — up to a common factor.

Once the conjecture is proved, the only candidates for flexible Kokotsakis-meshes are the four cases mentioned before. This should enable to classify of all flexible Kokotsakis-meshes.

5. Flexibility vs. reducibility of meshes

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