Rigidity Workshop, Toronto 2011

Pin Merging in Planar Body Frameworks

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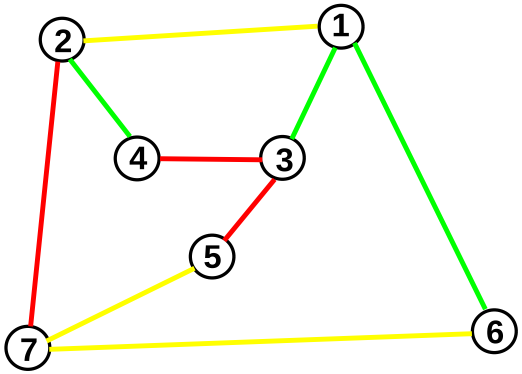
Meet a $\frac{3}{2}T$-graph

Meet a $\frac{3}{2}T$-graph



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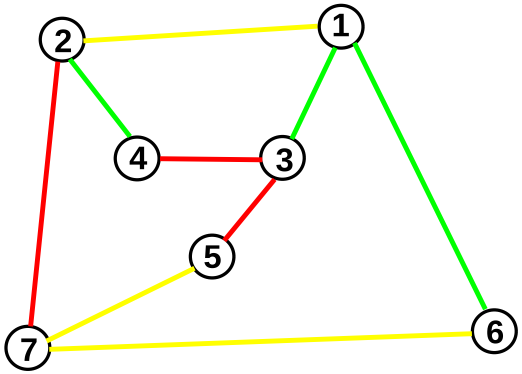

Definition:
$G = (V, E)$ is a $\frac{3}{2}T$-graph if

$$E = F_R \cup F_Y \cup F_G$$

with covering trees $F_R \cup F_Y$, $F_R \cup F_G$ and $F_Y \cup F_G$.

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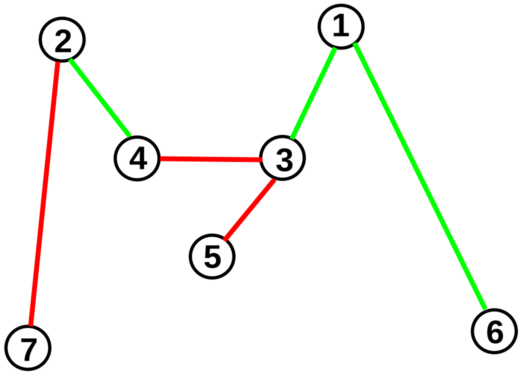

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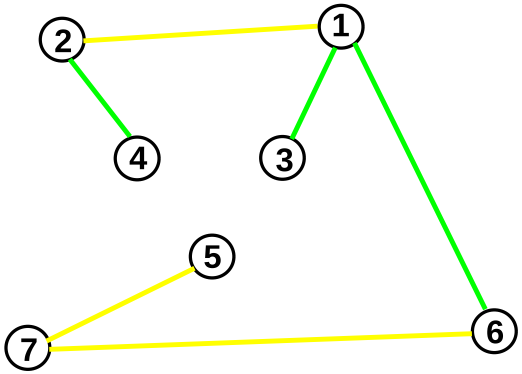

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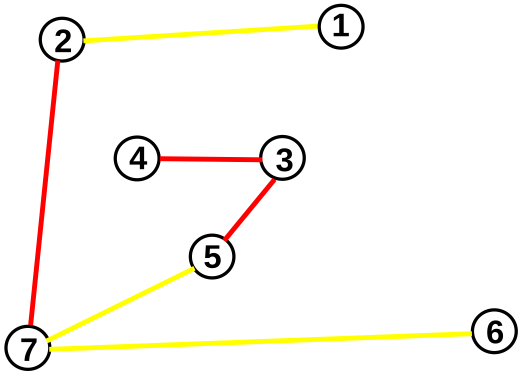

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(Indeed: $2G = T_{RY} \cup T_{RG} \cup T_{YG}$
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Conclusion: (Nash-Williams, Tutte)

$G = (V, E)$ is a $\frac{3}{2}T$-graph $\iff$
1. $2|E| = 3|V| - 3$
2. $\forall \emptyset \neq E' \subset E : \; 2|E'| \leq 3|V'| - 3$

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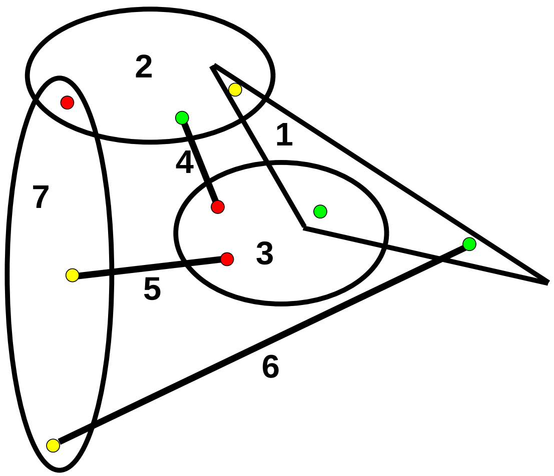

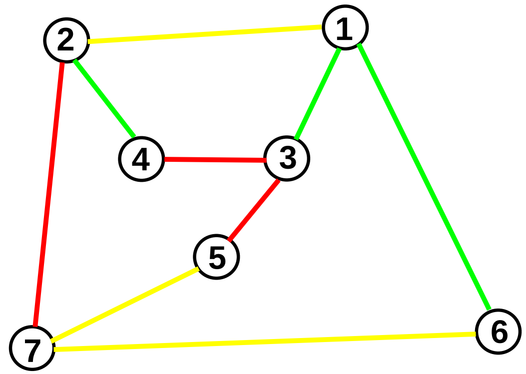

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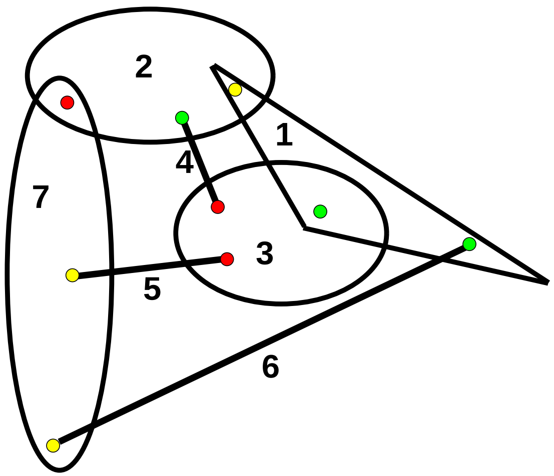

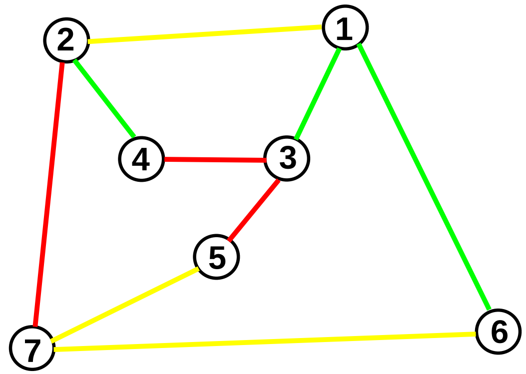

Remark: pins have degree 2 in generic realizations

Results in general dimensions

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Theorem: G can be realized as inf. rigid body-and-hinge framework in $I\!R^d$ iff. $(D-1)G$ contains D edge-disjoint spanning trees. (Tay-Whiteley)

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Special case ($d = 2$): $\frac{3}{2}T$-graph is a **minimal** design for inf. rigid body-and-pin framework in the plane.

Non-generic realizations

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Special case: $\frac{3}{2}T$-graph can be realized as inf. rigid frameworks in the plane with collinear pins for each body.
(Jackson-Jordán)

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General incidences: (allowing coinciding hinges = "multi-hinges")
count criterium on incidence graph $K_{b,h}$
but no tree decomposition
(Tay(?), 1987) (Tanigawa, 2011):

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(Tay(?), 1987) (Tanigawa, 2011):

$\exists$ rigid realization $\iff \exists\ I \subset (D-1)E(K_{b,h})$ s.t.

1) $|I| = D \cdot b + (D-1) \cdot h - D$
2) $\forall F \subset I$: $F \leq D \cdot B(F) + (D-1) \cdot H(F) - D$

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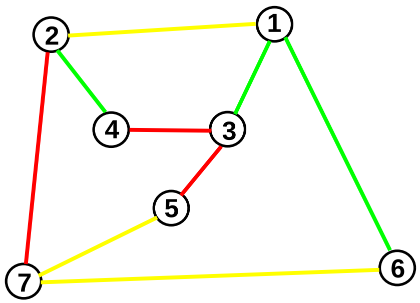

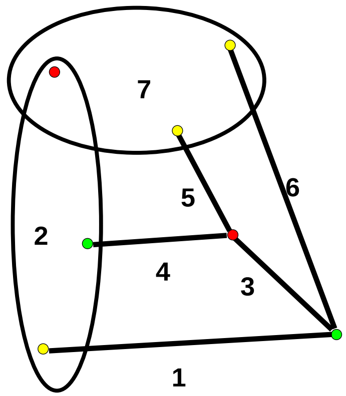

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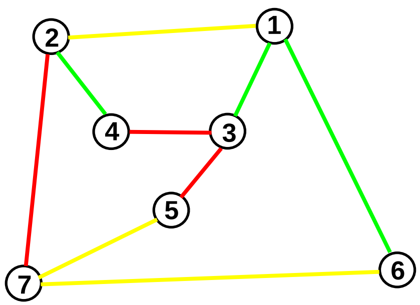
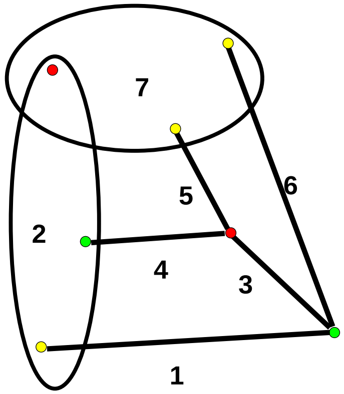




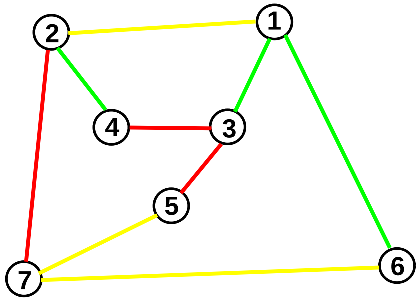
Question:

What pin mergings in $\frac{3}{2}T$-graphs preserve inf. rigidity?

Hypergraphs and merged pins

Example:

Hypergraphs and merged pins



constraint: bodies 1,2,4,7 attached by one pin

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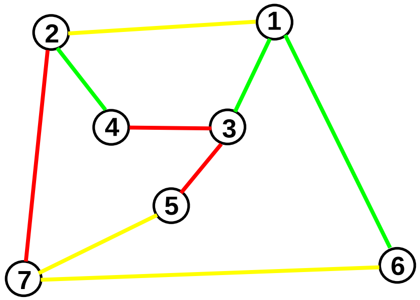

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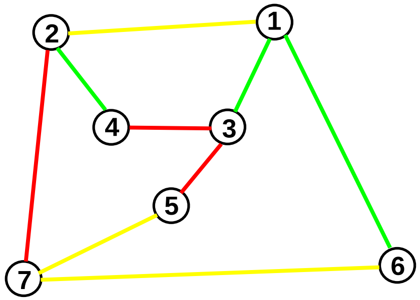

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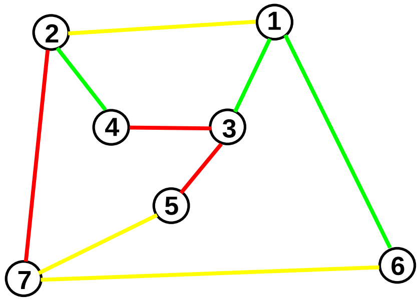

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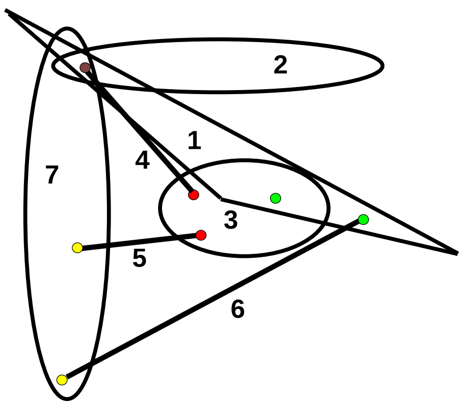

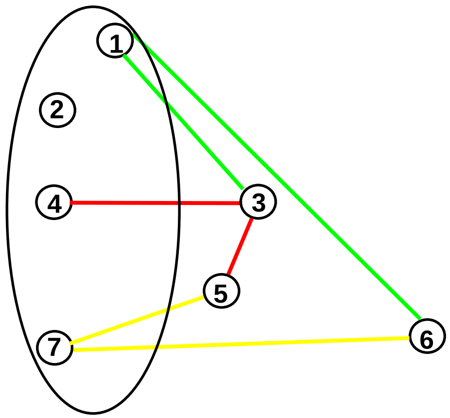

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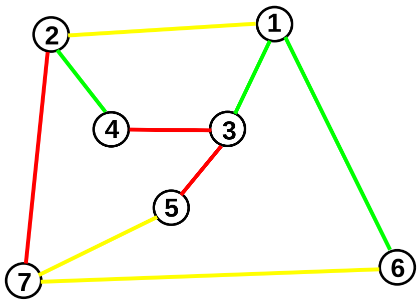

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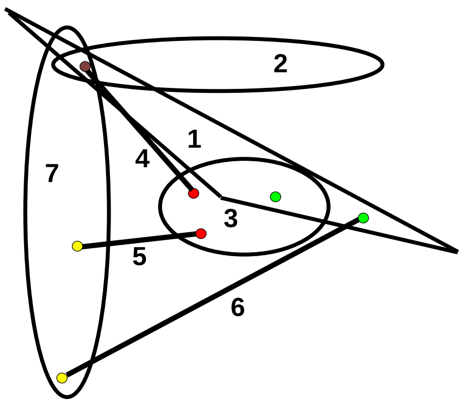

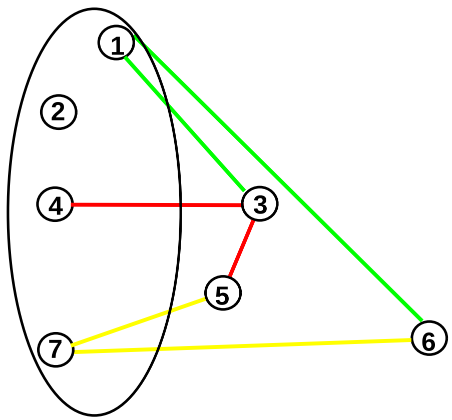

Notice: this pin merge causes non-trivial motions.

Intermezzo: weights of hyperedges

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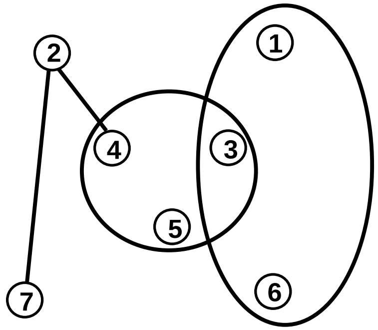

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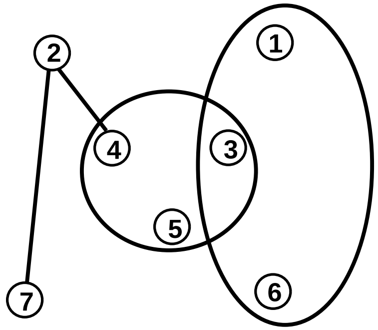

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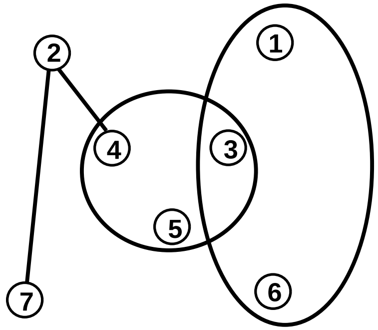

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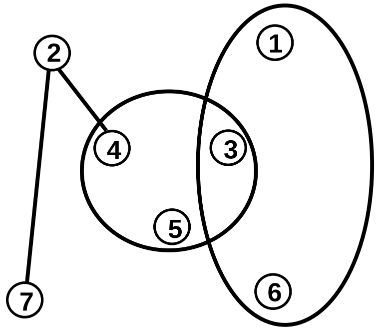

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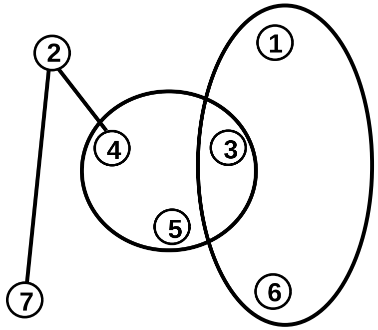

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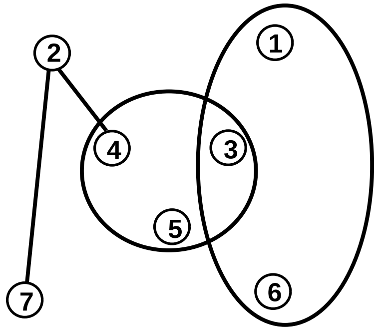

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G connected and no hypercycles

$\iff$ G connected and $w(E) = |V| - 1$

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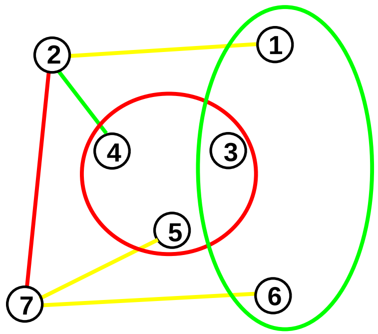
$\frac{3}{2}HT$-Hypergraphs



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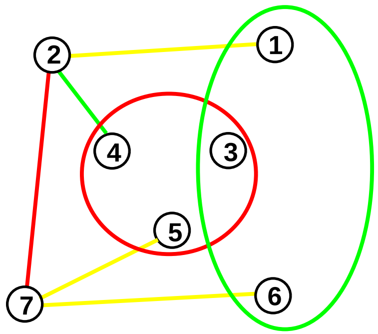

$\frac{3}{2}HT$-decomposition:

3 colours for hyperedges

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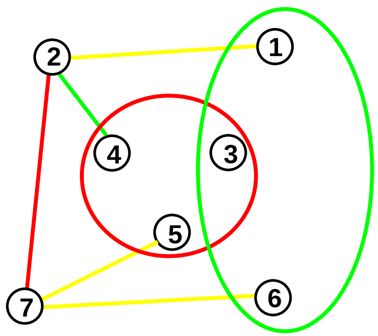

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union 2 colours $\longrightarrow$ **spanning hypertree**

$\frac{3}{2}HT$-Hypergraphs

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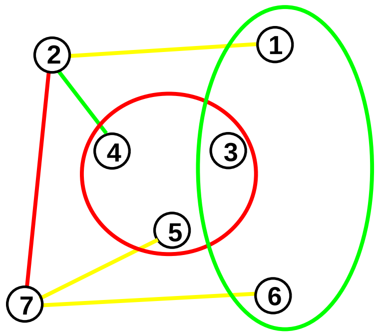

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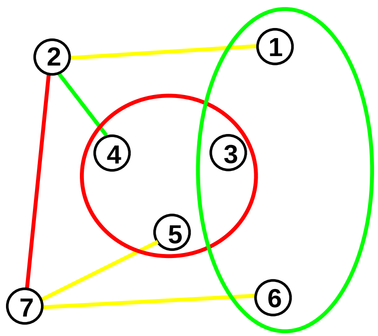

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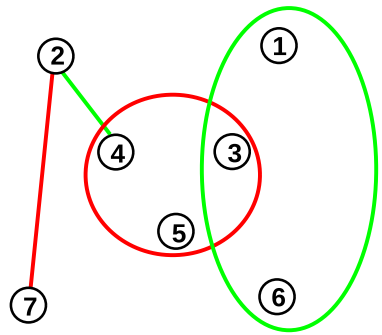

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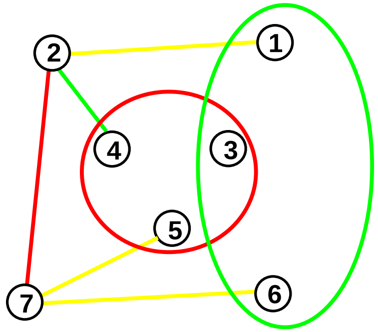

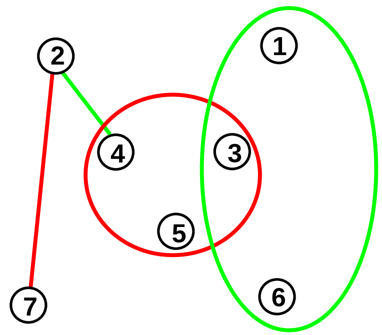

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Consequence: $\frac{3}{2}HT$-hypergraph $G = (V, E) \Rightarrow$

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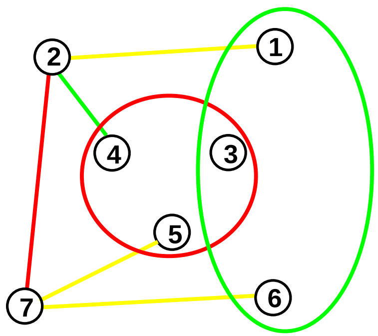

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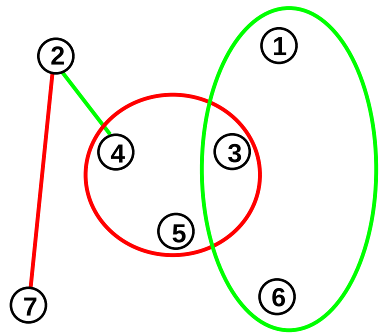

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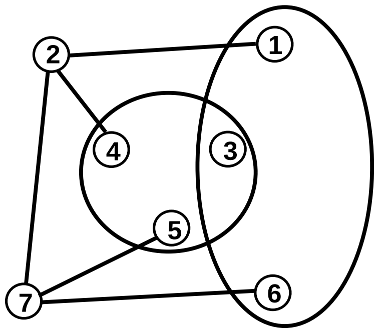
(3/2,3/2)-hypertight (?)

$\frac{3}{2}HT$ versus $\frac{3}{2}T$

hypergraph $G = (V, E)$ graph $G_2 = (V, E_2)$

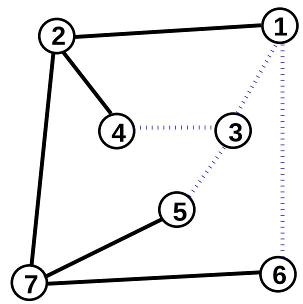


$\longrightarrow$

hosting

$\longleftarrow$

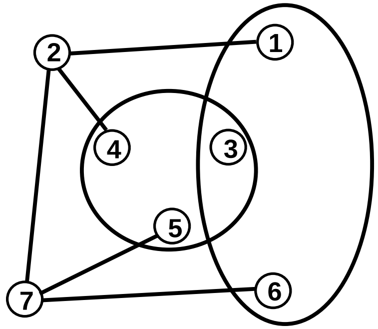
clustering



$\frac{3}{2}HT$ versus $\frac{3}{2}T$

hypergraph $G = (V, E)$

graph $G_2 = (V, E_2)$

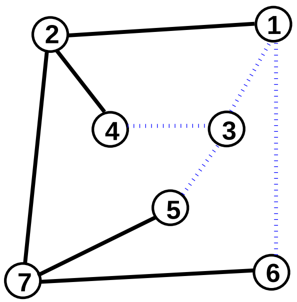


$\longrightarrow$

hosting

$\longleftarrow$

clustering



total weight $w(E)$

number of edges $|E_2|$

$\frac{3}{2}HT$ versus $\frac{3}{2}T$

hypergraph $G = (V, E)$

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$\longrightarrow$

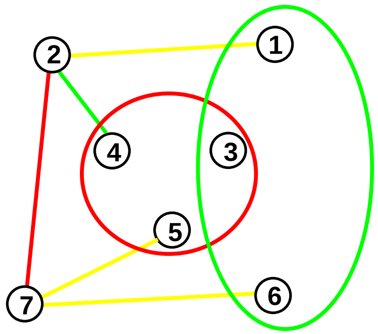
coloured

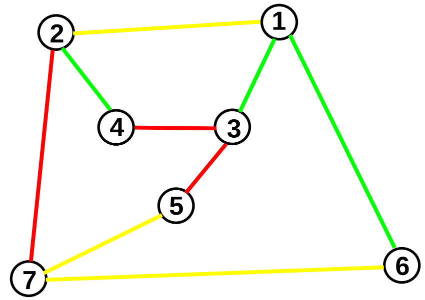
hosting

$\longleftarrow$

monochromatic

clustering





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$\longrightarrow$

always

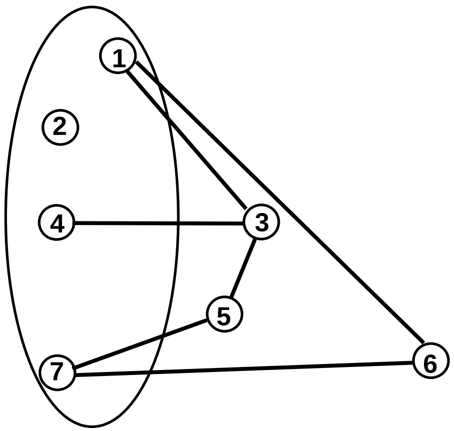
$\frac{3}{2}HT$-decomposition

$\frac{3}{2}T$-decomposition

$\longleftarrow$

sometimes

$\frac{3}{2}HT$-obstructions



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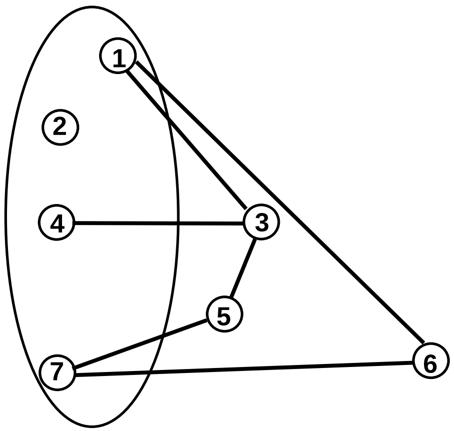

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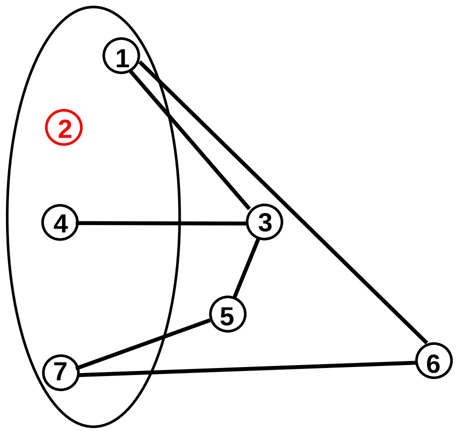

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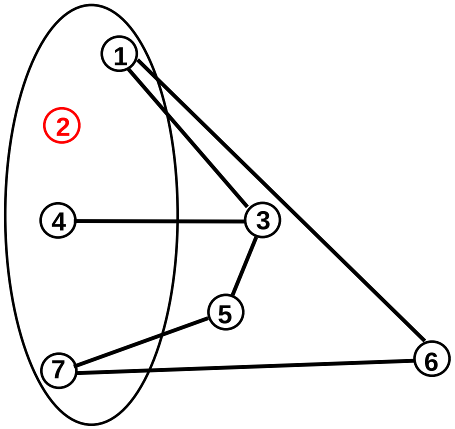

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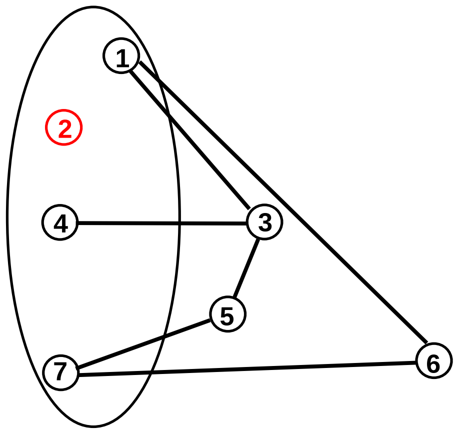

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Lucky guess: leaf-free + (D/(D-1),D/(D-1))-hypertight $\Longleftrightarrow$ $\frac{D}{D-1}HT$

$\frac{3}{2}HT$-Hypergraphs as rigid frameworks

Definition. Planar framework realization of hypergraph G = body-and-pin realization where hyperedges represent merged pins.

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The rigidity matrix I

Realization of hypergraph $G = (V, E)$ as body and pin framework in the plane: $F = (G, P)$

$$P: \; E \to \mathbb{R}^2 : \; e \mapsto P(e) = (x_e, y_e)$$

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For each edge $ij \in E_2$ with $\{i, j\} \subset e \in E$:

$$\begin{aligned} JX_{ij} &= (0, \ldots, 0, 1, 0 \ldots, 0, -1, 0, \ldots, 0|0, \ldots, 0|0, \ldots, 0, -x_e, 0, \ldots, 0, x_e, 0, \ldots, 0) \\ JY_{ij} &= (0, \ldots, 0|0, \ldots, 0, 1, 0 \ldots, 0, -1, 0, \ldots, 0|0, \ldots, 0, -y_e, 0, \ldots, 0, y_e, 0, \ldots, 0) \end{aligned}$$

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$\Rightarrow 2w(E) \times |V|$ matrix $M(G_2, P)$.

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Remarks:

- rank $M(G_2, P)$ independent from host
- F inf. rigid $\iff$ rank $M(G_2, P) = 3|V| - 3$
- F isostatic $\iff M(G_2, P)$ has independent rows and $2 \cdot w(E) = 3|V| - 3$

Independent hypergraphs

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Remark. Our count is equivalent to the Tay-Tanigawa criterion for $d = 2$ (extra condition: no leaves).

Results and conjectures: overview

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(conjectured)

Matrix proof

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variables $(\mathbf{X}, \mathbf{Y}) = (X_e, Y_e, \ldots)$ for each hyperedge e

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(doubled) edges hosting the same hyperedge belong to the same trees

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Rearrange rows of $M(G_2, \mathbf{X}, \mathbf{Y})$: ($T_2 = T_{2x} \cup T_{2y}$)

$$\left(\begin{array}{c|c|c} I(T_1) & 0(T_1) & X(T_1) \\ I(T_{2x}) & 0(T_{2x}) & X(T_{2x}) \\ 0(T_{2y}) & I(T_{2y}) & Y(T_{2y}) \\ 0(T_3) & I(T_3) & Y(T_3) \end{array}\right)$$

Matrix proof (continued)

Specialization $(\mathbf{X}, \mathbf{Y})$:

$$
\begin{aligned}
e \text{ covered by } T_1 &\Rightarrow X_e = 0\\
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Theorem: Assume no leaves. G is d-independent iff.
$\forall \emptyset \neq E' \subset E : \ (D-1) \cdot w(E') \leq D|\cup E'| - D$.
Proof. Tay-Tanigawa count for rigidity.

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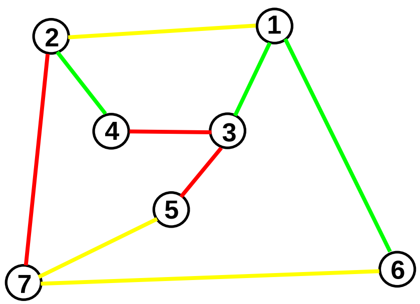

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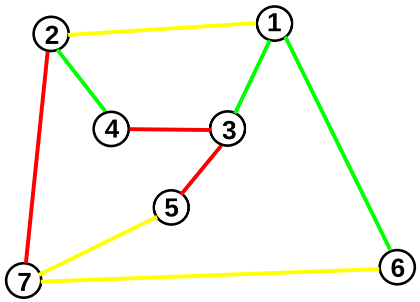

Question: How do we find other decompositions?

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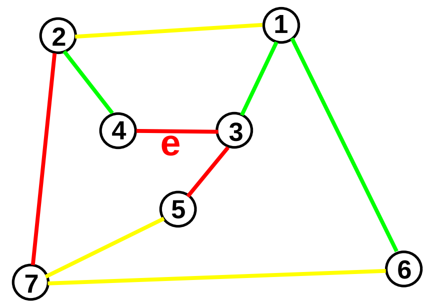

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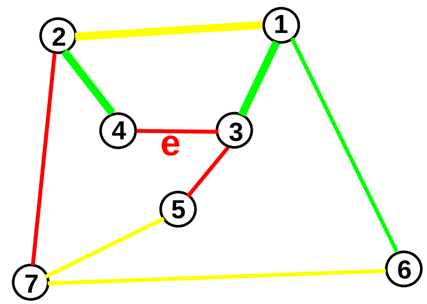

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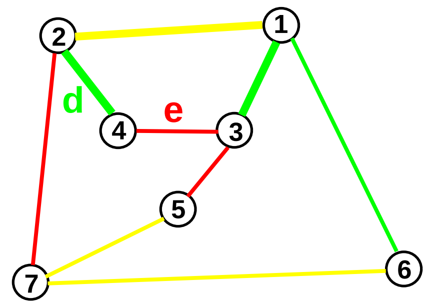

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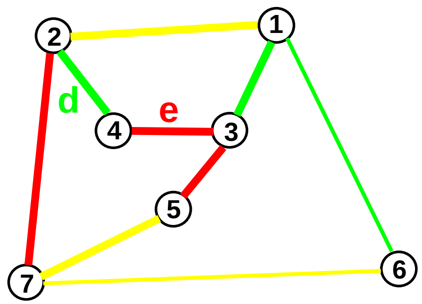

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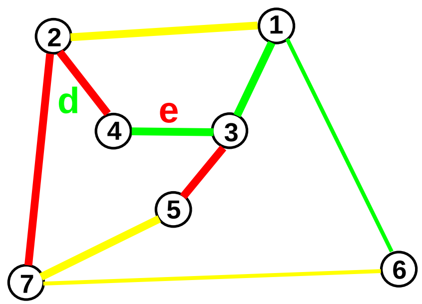

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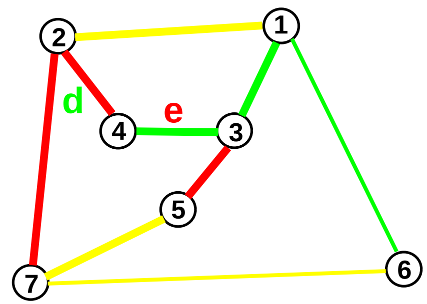

Observe: $T_{YG} + e_1 - e_2$ and $T_{RY} + e_2 - e_1$ still trees!

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$e \in E \to \gamma(e)$: unique path in G without colour(e) connecting endpoints e

Colour swap condition

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QED

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- Generalization to spatial body-pin frameworks? (allowing multi-pins, and body pairs sharing 2 pins)

Any answers?