The Brahmahupta's theorem after Coxeter

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History

Heron of Alexandria (60 B.C.) left us the following remarkable formula that relates the area $\mathcal{A}$ of a triangle to its side lengths a, b and c

$$A = \sqrt{(s-a)(s-b)(s-c)s},$$

where $s = (a+b+c)/2$ is the semiperimeter.
Brahmagupta (XVII century) gave the analogues formula for a convex cyclic (= inscribed in a circle) quadrilateral with side lengths a, b, c and d

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)},$$

where $s = (a+b+c+d)/2$.
D.P. Robbins (1994) found a way to generalize these formulas.

History

The general result is the following

Theorem 1 (Robbins, 1994)

For each $n \geq 3$ there is a unique irreducible homogeneous polynomial α_n with integer coefficients, such that

$$\alpha_n(16A^2, a_1^2, \ldots, a_n^2) = 0,$$

whenever $a_1, \ldots, a_n$ are side lengths of a cyclic $n-$gon and A is its area.

The polynomials α_n are known in the literature as generalized Heron polynomials. Certainly, the Heron's and Brahmagupta's theorems are the partial cases of the above theorem. The properties of polynomials α_n were investigated by V.V. Varfolomeev (2003) and M. Fedorchuk and I. Pak (2005). Related results are also obtained by Ren Guo and Nilgün Sönmez (2010).

History

A three dimensional version of the Heron formula belongs to Tartaglia (1499-1557) who found a formula for the volume of Euclidean tetrahedron. More precisely, let be an Euclidean tetrahedron with edge lengths $d_{ij},\ 1 \le i < j \le 4$. Then $V = \mathrm{Vol}(T)$ is given by

$$288V^2 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & d_{12}^2 & d_{13}^2 & d_{14}^2 \\ 1 & d_{21}^2 & 0 & d_{23}^2 & d_{24}^2 \\ 1 & d_{31}^2 & d_{32}^2 & 0 & d_{34}^2 \\ 1 & d_{41}^2 & d_{42}^2 & d_{43}^2 & 0 \end{vmatrix}.$$

Note that V is a root of quadratic equation whose coefficients are integer polynomials in $d_{ij},\ 1 \le i < j \le 4$. High dimensional generalization of this result is known as the Cayley-Menger formula.

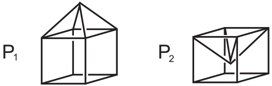
History

Surprisedly, but the result of Tartaglia can be generalized on any Euclidean polyhedron in the following way.

Theorem 2 (I.H. Sabitov, 1996)

Let P be a simplicial Euclidean polyhedron. Then $V = Vol(P)$ is a root of an even degree algebraic equation whose coefficients are integer polynomials in edge lengths of P depending on combinatorial type of P only.

Example



(All edge lengths are taken to be 1)

The volumes $V_1 = Vol(P_1)$ and $V_2 = Vol(P_2)$ are roots of the same algebraic equation $a_0 V^{2n} + a_1 V^{2n-2} + \ldots + a_n V^0 = 0$.
Recently, A.A. Gaifullin (2011) proved a four dimensional version of the Sabitov's theorem.

History

Non-Euclidean versions of the Robbins and Sabitov theorems are not known yet. In the same time the volume of non-Euclidean tetrahedron was investigated by many authors.
A formula the volume of an arbitrary hyperbolic tetrahedron has been unknown until recently. The general algorithm for obtaining such a formula was indicated by W.–Y. Hsiang (1988) and the complete solution of the problem was given by Yu. Cho and H. Kim (1999), J. Murakami, M. Yano (2001) and A. Ushijima (2002).
An excellent exposition of these results and a complete geometric proof of the volume formula was given by Y. Mohanty (2003) in her Ph.D. thesis.
A simple integral formula was obtained in our joint paper D. Derevnin and A. Mednykh (2005).
More than a century ago, in 1906, the Italian mathematician G. Sforza found the formula for the volume of a non-Euclidean tetrahedron. It was discovered during a discussion of the author with J. M. Montesinos in August 2006.

Non-Euclidean geometry

We start with the following well-known results from non-Euclidean geometry. The area of a triangle with angles α, β and γ is given by the formulas

- $A = \pi - \alpha - \beta - \gamma, \qquad (\mathbb{H}^2)$
- $A = \alpha + \beta + \gamma - \pi, \qquad (\mathbb{S}^2)$
- $A = s^2 \tan(\alpha/2) \tan(\beta/2) \tan(\gamma/2). \qquad (\mathbb{E}^2)$

In the later formula the semiperimeter s plays a role of scale on the Euclidean plane $\mathbb{E}^2$.

Non-Euclidean geometry

There are three non-Euclidean version of the Heron formula on the hyperbolic plane. The area A of a hyperbolic triangle with side lengths $a, b,$ and c is given by each of the following formulas

- Sine of 1/2 Area Formula

$$\sin^2 \frac{A}{2} = \frac{\sinh(s-a)\sinh(s-b)\sinh(s-c)\sinh(s)}{4\cosh^2\left(\frac{a}{2}\right)\cosh^2\left(\frac{b}{2}\right)\cosh^2\left(\frac{c}{2}\right)},$$

- Tangent of 1/4 Area Formula

$$\tan^2 \frac{A}{4} = \tanh(\frac{s-a}{2})\tanh(\frac{s-b}{2})\tanh(\frac{s-c}{2})\tanh(\frac{s}{2}),$$

- Sine of 1/4 Area Formula

$$\sin^2 \frac{A}{4} = \frac{\sinh(\frac{s-a}{2})\sinh(\frac{s-b}{2})\sinh(\frac{s-c}{2})\sinh(\frac{s}{2})}{\cosh\left(\frac{a}{2}\right)\cosh\left(\frac{b}{2}\right)\cosh\left(\frac{c}{2}\right)}.$$

The third formula can be obtained by the squaring of the product of the first two.

Brahmagupta's theorem for cyclic non-Euclidean quadrilateral

Theorem 3 (Sine of 1/2 Area Formula, M., 2011)

The area A of a cyclic hyperbolic quadrilateral with side lengths a, b, c and d can be found by the formula

$$\sin^2\frac{A}{2}=\frac{\sinh(s-a)\sinh(s-b)\sinh(s-c)\sinh(s-d)}{4\cosh^2\left(\frac{a}{2}\right)\cosh^2\left(\frac{b}{2}\right)\cosh^2\left(\frac{c}{2}\right)\cosh^2\left(\frac{d}{2}\right)}(1-\varepsilon),$$

where

$$\varepsilon=\frac{\sinh(\frac{a}{2})\sinh(\frac{b}{2})\sinh(\frac{c}{2})\sinh(\frac{d}{2})}{\cosh(\frac{s-a}{2})\cosh(\frac{s-b}{2})\cosh(\frac{s-c}{2})\cosh(\frac{s-d}{2})}$$

and $s=(a+b+c+d)/2$.

We note that if $d=0$ then $\varepsilon=0$ and the theorem reduces to the correspondent theorem for a hyperbolic triangle.

Brahmagupta's theorem for cyclic non-Euclidean quadrilateral

Theorem 4 (Tangent of 1/4 Area Formula, M., 2011)

The area A of a cyclic hyperbolic quadrilateral with side lengths a, b, c and d can be found by the formula

$$\tan^2 \frac{A}{4} = \frac{1}{1-\varepsilon} \tanh(\frac{s-a}{2}) \tanh(\frac{s-b}{2}) \tanh(\frac{s-c}{2}) \tanh(\frac{s-d}{2}),$$

where

$$\varepsilon = \frac{\sinh(\frac{a}{2}) \sinh(\frac{b}{2}) \sinh(\frac{c}{2}) \sinh(\frac{d}{2})}{\cosh(\frac{s-a}{2}) \cosh(\frac{s-b}{2}) \cosh(\frac{s-c}{2}) \cosh(\frac{s-d}{2})}$$

and $s = (a+b+c+d)/2$.

If $d = 0$ then $\varepsilon = 0$ and the theorem reduces to the theorem for a hyperbolic triangle.

Brahmagupta's theorem for cyclic non-Euclidean quadrilateral

By squaring the product of the two previous area formulas we obtain

Theorem 5 (Sine of 1/4 Area Formula, M., 2011)

The area A of a cyclic hyperbolic quadrilateral with side lengths a, b, c and d can be found by the formula

$$\sin^2 \frac{A}{4} = \frac{\sinh(\frac{s-a}{2})\sinh(\frac{s-b}{2})\sinh(\frac{s-c}{2})\sinh(\frac{s-d}{2})}{\cosh\left(\frac{a}{2}\right)\cosh\left(\frac{b}{2}\right)\cosh\left(\frac{c}{2}\right)\cosh\left(\frac{d}{2}\right)},$$

where $s = (a + b + c + d)/2.$

Brahmagupta's theorem for cyclic non-Euclidean quadrilateral

As an immediate consequence of the above mentioned theorems we have the following corollary.

Corollary 1

For any cyclic hyperbolic quadrilateral the following inequalities take a place

$$\sin^2\frac{A}{2} < \frac{\sinh(s-a)\sinh(s-b)\sinh(s-c)\sinh(s-d)}{4\cosh^2\left(\frac{a}{2}\right)\cosh^2\left(\frac{b}{2}\right)\cosh^2\left(\frac{c}{2}\right)\cosh^2\left(\frac{d}{2}\right)}$$

and

$$\tan^2\frac{A}{4} > \tanh\left(\frac{s-a}{2}\right)\tanh\left(\frac{s-b}{2}\right)\tanh\left(\frac{s-c}{2}\right)\tanh\left(\frac{s-d}{2}\right).$$

Brahmagupta's theorem for inscribed and circumscribed quadrilateral

Corollary 2

The area A of a cyclic (=inscribed) and circumscribed hyperbolic quadrilateral with side lengths a, b, c and d can be found by the formula

$$\sin^2\frac{A}{4}=\tanh(\frac{a}{2})\tanh(\frac{b}{2})\tanh(\frac{c}{2})\tanh(\frac{d}{2}).$$

An Euclidean version of this result is known for a long time. See for example (Ivanoff, 1960). In this case

$$A^2=a\,b\,c\,d.$$

Sketch of the proof

The proof is based on the following two observations.

1° The necessary and sufficient condition for hyperbolic quadrilateral to be inscribed into circle, horocycle or one branch of an equidistant curve were suggested by J. E. Valentine (1970) who were influenced by H. S. M. Coxeter. In terms of side lengths it can be given by the following the non-Euclidean version of Ptolemy's theorem.

$$s(a)s(c) + s(b)s(d) = s(e)s(f),$$

where $s(x) = \sinh(\frac{x}{2})$, and e and f are lengths of the diagonals.

2° The necessary and sufficient condition for hyperbolic quadrilateral to be inscribed into circle, horocycle or one branch of an equidistant curve were given by F.V. Petrov (2009) in terms of angles. They are just

$$A + C = B + D.$$