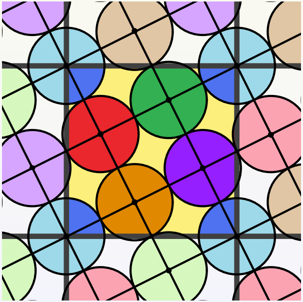
Packings of Equal Circles on Flat Tori

William Dickinson
Workshop on Rigidity
Fields Institute
October 14, 2011



Introduction

Goal

Understand locally and globally maximally dense packings of equal circles on a fixed torus.

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Which Torus?

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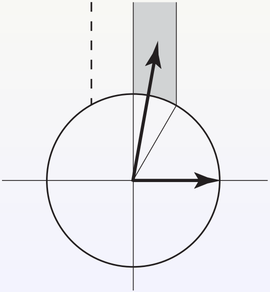

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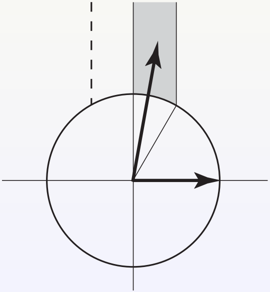

For the optimal packings of 2 circles on any torus with a length one closed geodesic see the work of Przeworski (2006).

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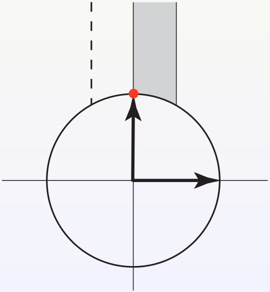

A Square Torus is the quotient of the plane by unit perpendicular vectors. See the work of H. Mellisen (1997) – proofs for 3 and 4 circles and conjectures up to 19 circles. For large numbers (> 50) see the work of Lubachevsky, Graham, and Stillinger (1997).

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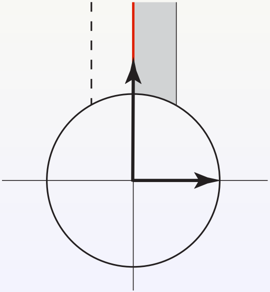

A Rectangular Torus is the quotient of the plane by perpendicular vectors. See the work of A. Heppes (1999) – proofs for 3 and 4 circles.

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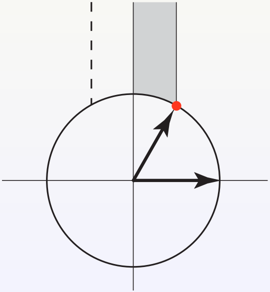
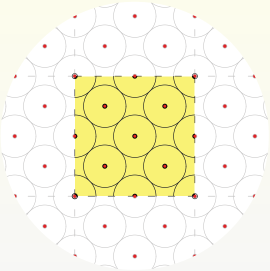


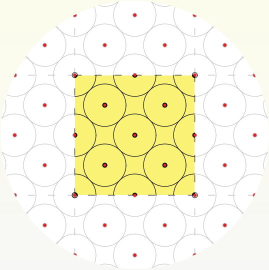
A Triangular Torus is the quotient of the plane by unit vectors with a 60 degree angle between them. Understanding packings on this torus might help prove a conjecture of L. Fejes Tóth on the solidity of the triangular close packing in the plane with one circle removed.

Packing Graphs & Strut Frameworks

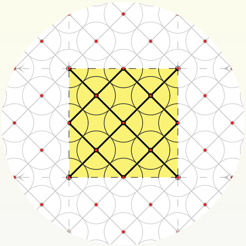


Circle Packing

Packing Graphs & Strut Frameworks



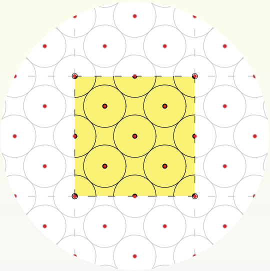
$\Leftrightarrow$



Circle Packing

Equilateral Toroidal Packing Graph

Packing Graphs & Strut Frameworks



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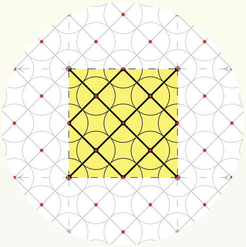
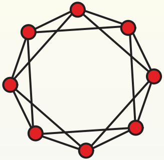


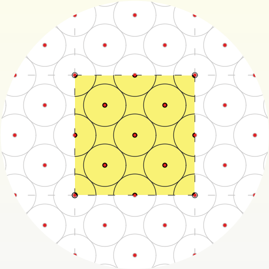
Equilateral Toroidal Packing Graph

$\Rightarrow$



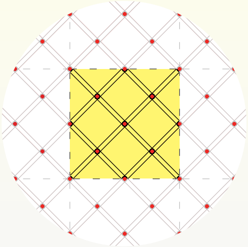
Combinatorial Graph

Packing Graphs & Strut Frameworks

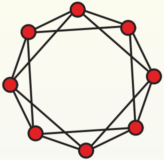


Circle Packing

⇔



Equilateral Toroidal Strut Framework

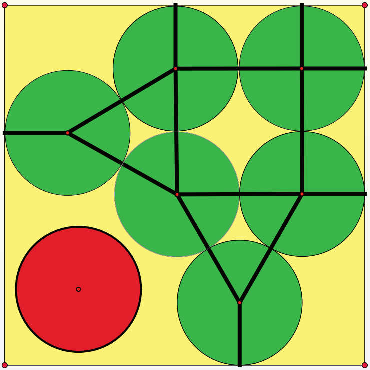


Combinatorial Graph

Viewing the packing graph as a strut framework helps us understand the possible combinatorial (multi-)graphs.

Rigid Spine And Free Circles

Consider the optimal packing of seven circles a hard boundary square. Due to Schear/Graham(1965) Mellisen(1997)



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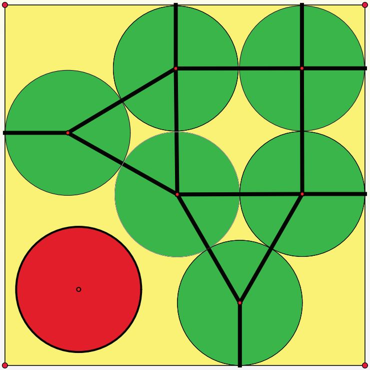

The red circle is a *free circle* and the packing graph associated to the green circles form the *rigid spine*.

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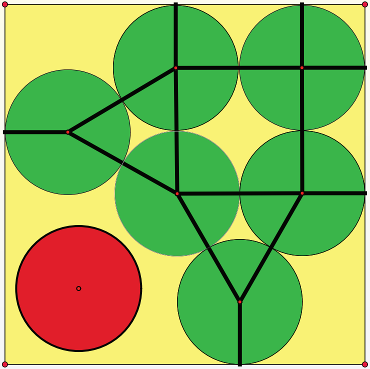

The red circle is a *free circle* and the packing graph associated to the green circles form the *rigid spine*. In what follows we will only consider packings without free circles.

Strut Frameworks: Rigidity and Infinitesimal Rigidity

An assignment of vectors $(\vec{p}_1, \vec{p}_2, \vec{p}_3, \ldots, \vec{p}_n)$ to each of the vertices $(p_1, p_2, p_3, \ldots, p_n)$ in a toroidal strut framework is a *infinitesimal flex* of the arrangement if

$$(\vec{p}_i - \vec{p}_j) \cdot (p_i - p_j) \geq 0$$

for each strut (i, j) in the framework.

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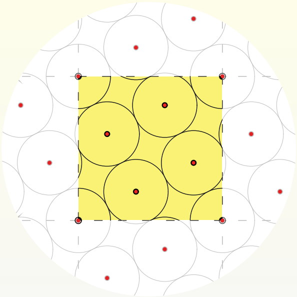
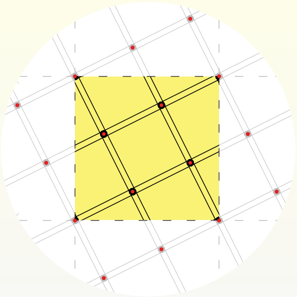
Theorem (Connelly)

A (toroidal) strut framework is (locally) rigid if and only if infinitesimally rigid

Optimal Arrangements and Toroidal Strut Frameworks





Observation

Given a packing, if the associated toroidal strut framework is (locally) rigid then the packing is locally maximally dense.

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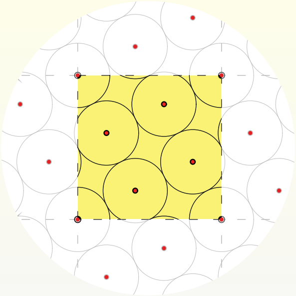

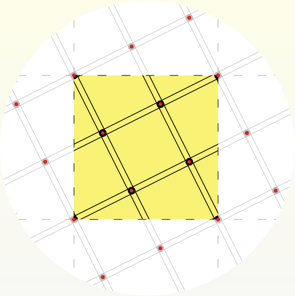

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Theorem (Connelly)

If a toroidal packing is locally maximally dense then there is a subpacking whose associated toroidal strut framework is (locally) rigid.

Combinatorial Graph Edge Restrictions

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A locally maximally dense packing of n circles on a flat torus (without free circles) has at least $2n - 1$ edges.

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Combinatorial Graph Edge Restrictions

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  - → A combinatorial graph associated to an optimal packing has between $2n - 1$ and $3n$ edges.

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- To be infinitesimally rigid each circle must be tangent to at least 3 others and the points of tangency can't be restricted to a closed semi-circle.

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  - $\rightarrow$ Every vertex in a combinatorial graph associated to an optimal packing is incident to between 3 and 6 edges.

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  - → Every vertex in a combinatorial graph associated to an optimal packing is incident to between 3 and 6 edges.

Note: These observations are enough to determine all the optimal packings of 1-4 circles on a square flat torus.

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To determine all possible optimal packings of n circles on a torus:

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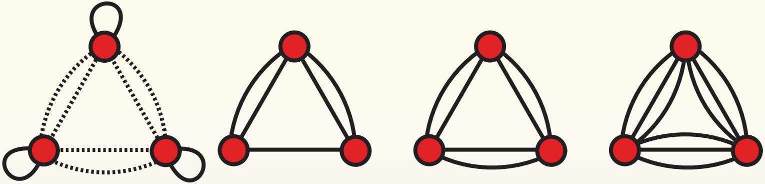
4. Determine which equilateral embeddings are associated to locally maximally dense packings.

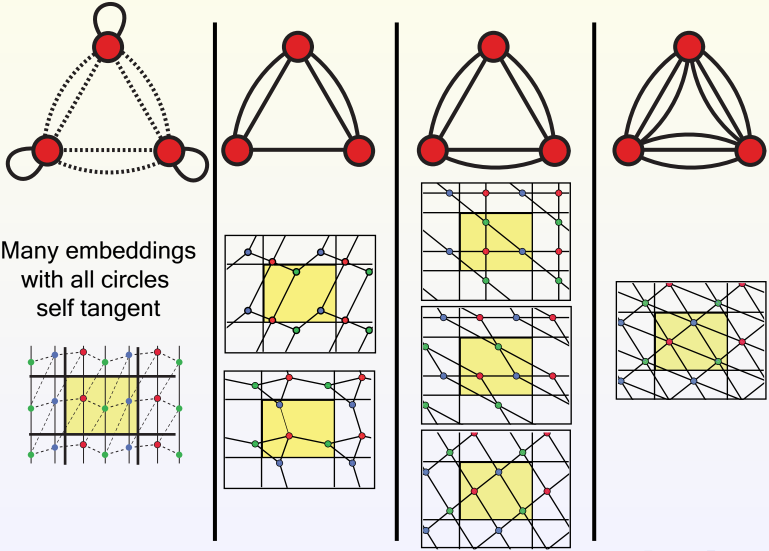
Three Circle Case

Step 1: Partial list of possible combinatorial graphs.



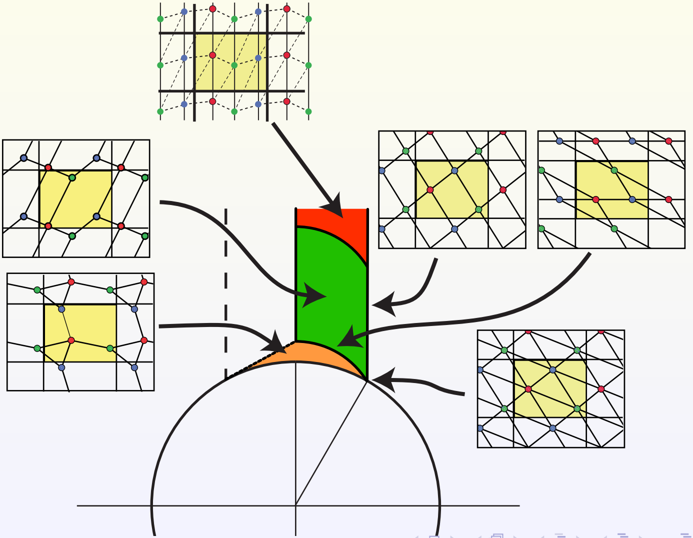
Three Circle Case

Step 2: Partial list of all embeddings of the combinatorial graphs on a topological torus.

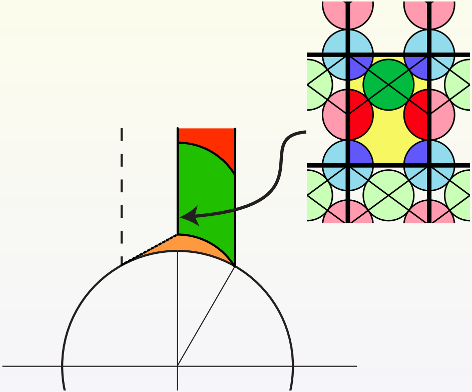


Three Circle Case

Steps 3 & 4: Equilateral Embeddings and Locally Maximally Dense Packing/Regions.



Minimally Dense Arrangements



Rectangular Torus, ≈ 1.35 ratio

Density $= \frac{2\pi\sqrt{3}}{\sqrt{138+22\sqrt{33}}} \approx 0.66930$

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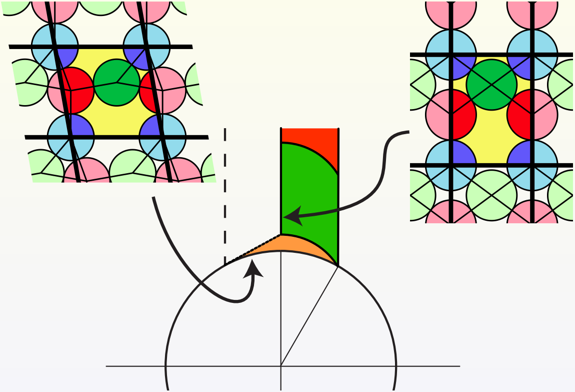

Equilateral Torus, 100 Degrees

Density $= \frac{3\pi}{16\sin(\frac{4\pi}{9})} \approx 0.61673$

Rectangular Torus, ≈ 1.35 ratio

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Tool: Stressed Arrangements

Definition

A collection of scalars $\omega_{ij} = \omega_{ji}$ (one for each strut) is called an *self-stress* if $\sum_j \omega_{ij}(p_j - p_i) = \vec{0}$ for all vertices p_i.

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Theorem (Roth-Whiteley)

A toroidal strut framework is (infinitesimally) rigid if and only if it is infinitesimally rigid as a bar framework and it has a self-stress that has the same sign and is non-zero on every strut.

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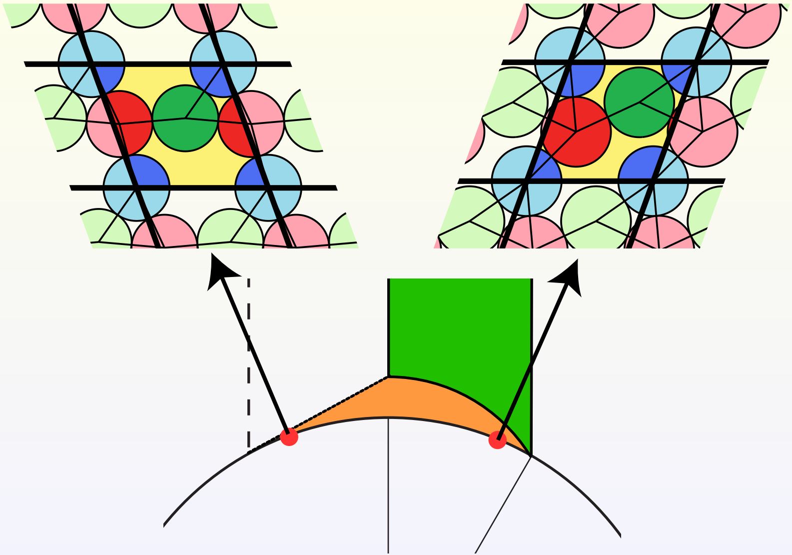
Theorem (Connelly)

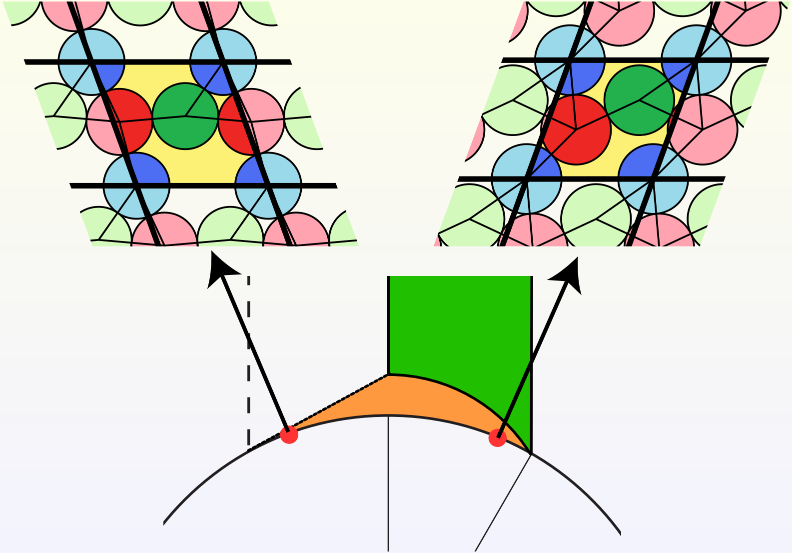
On a fixed torus, suppose there is a packing so that the associated equilateral strut framework, F, is infinitesimally rigid then any other infinitesimally rigid, equilateral strut framework freely homotopic to F on the torus is congruent to F by translation.

Two Locally Optimally Dense Arrangements on One Torus



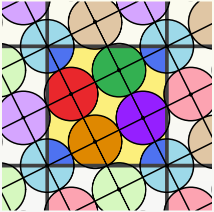
Two Locally Optimally Dense Arrangements on One Torus

The packing graphs are not homotopic on the fixed torus.



Other Results on the Square and Triangular Torus

Using the same techniques the following are optimally dense.



5 Circles
Square Torus
10 contacts

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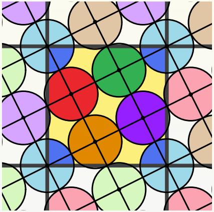

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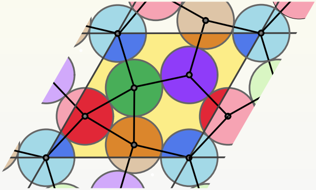

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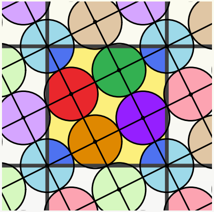

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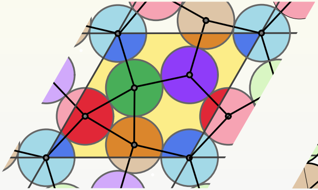

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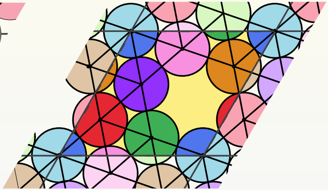

6 Circles
Triangular Torus
18 contacts

Strictly Jammed / Periodically Stable Packings

Definition

A packing on a torus is *strictly jammed* if there is no non-trivial infinitesimal motion of the packing, as well as the lattice defining the torus, subject to the condition that the total area of the torus does not infinitesimally increase.

Strictly Jammed / Periodically Stable Packings

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A packing on a torus is *strictly jammed* if there is no non-trivial infinitesimal motion of the packing, as well as the lattice defining the torus, subject to the condition that the total area of the torus does not infinitesimally increase.

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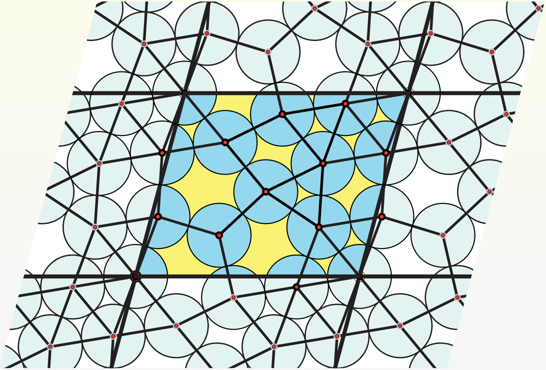
To have a unique solution, you must have one more inequality/constraint than unconstrained variables so $(e+1) \geq (2n+4) - (2+1) + 1$ or

$$e \geq 2n + 1$$

in order to possibly be strictly jammed.

Non-Triangular-Close Based Strictly Jammed Example (Connelly)



10 Circles

$\approx 75^\circ$ Torus with ≈ 1.17 Ratio

22 contacts

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- Is there a 3-d analog for this algorithm for packing sphere in a 3-torus?

Thank You

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