Day 4

6/30/11

Colorings and Homomorphisms for Graphs and Finite Structures

Jaroslav Nesetril

Fiber Construction $G \rightsquigarrow G * M \in CSP(H)$

M is 3-partitionable

Defn H is block projective if $\exists S$, $|S| \geq 2$, $S = \{0,1\}$
$\forall s$ $H_s \subseteq V(H)$ $H_s \cap H_{s'} = \phi$ $\phi : H^d \to H$ $\exists i$
$(s_1, \dots, s_d) \in S^d$ $\phi(s_1, \dots, s_d) \in H_{s_i}$

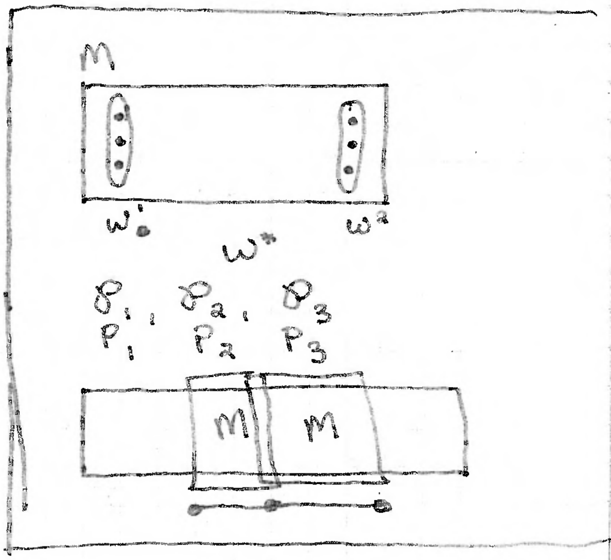
Lemma If H idempotent relational core which is block projective Then H is K_3 partitionable and thus $CSP(H)$ is NP-complete

$S = \{0,1\}$ H_0, H_1 $G = H^6$
$\omega^1 = \{001111, 110011, 111100\}$
$\omega^2 = \{110101, 011110, 101011\}$

$\rho_1 = \{(x, y_1, y_2) \mid x \in H_0, y_i \in H_1\}$

$\rho_2 = \{(y_1, x, y_2) \mid x \in H_0, y_i \in H_1\}$

$\rho_3 = \{(y_1, y_2, x) \mid x \in H_0, y_i \in H_1\}$



Thm (N., Siggers, Zadori) H idempotent core
TFAE
① H is K_3 partitionable
② H is block projective
Alg. conj [③ A_H has a factor whose term operations are projections] ?

$CSP(H) = \{G \mid G \to H\} = \to H$
membership problem for this class.

When is $CSP(H)$ first order (FO) definable?

$\phi(\forall, \exists, x, y_i, e, R)$ FO

$CSP(H) = \to H$ $\overline{CSP(H)} = \{A \mid A \not\to H\}$.

notes by Megan Heenehan

Observation ③ $\overline{CSP(H)}$ is closed on homomorphism

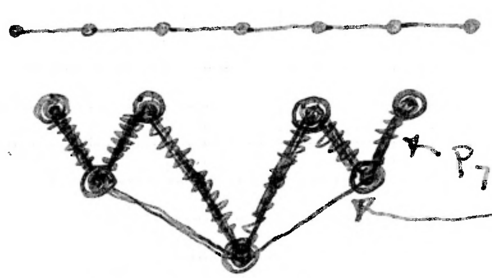
$A \in \overline{CSP(H)}$ $\quad A \to B$ then $B \in \overline{CSP(H)}$

Thm (Rossman '05) $\mathcal{C}$ is a class of relational structures closed on homomorphism and which is FO definable $\Leftrightarrow$ FO^{+} definable
$\Leftrightarrow \exists\ \mathcal{F}$ finite set structure s.t.
$\mathcal{C} = \mathcal{F}^{\rightarrow} = \{A \mid \exists F \in \mathcal{F}\ \ F \to A\}$

Homomorphism Order

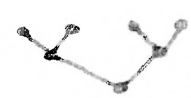
quantifier depth $\approx$ tree depth of graph

Defn The tree depth of a graph G
$td(G) =$ min height of rooted forest T such that $G \subseteq$ closure (T).



$td(P_7) \leq 7$

$td(P_7) = 3$.

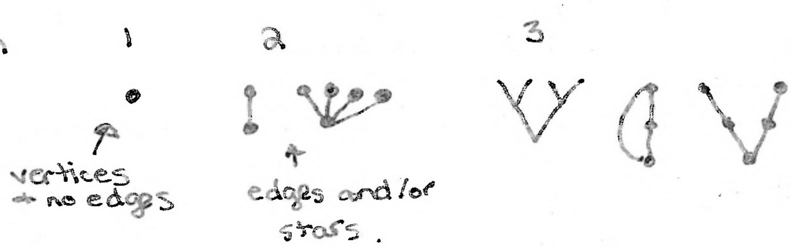
(underlying tree )

$tw(G) \leq td(G) \leq tw \cdot \log|V(G)|$

tree width

Thm $\forall\ d$ $\quad \{A \mid td(A) \leq d\}$ has finitely many cores.

Treedepth



If CSP(H) is FO def. then $\exists$ $\mathcal{F}$ - finite
$\underbrace{CSP(H) = Forb(\mathcal{F})}_{\text{homomorphism duality}} = \{A \mid F \in \mathcal{F} \Rightarrow F \not\to A\}$

(triangle free graphs)

 $\not\to G \Leftrightarrow G \to \bullet$ only duality that exists for undirected graphs.

Examples for oriented graphs

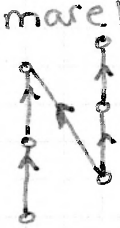
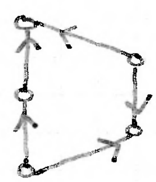
 $\not\to G \Leftrightarrow G \to \bullet$

P_{k+1}  $\not\to G \Leftrightarrow G \to$  T_k

Gallai, Roy, M. Hasse, Vitkaver

$\chi(G) > k \Leftrightarrow$ every orientation of $G \supseteq \vec{P}_{k+1}$

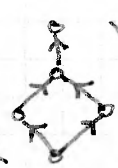
Komarek

 $\not\to G \Leftrightarrow G \to$ 

Thm (Komarek, N., Tardiff 2000 in JCTB) For a set $\mathcal{F}$ of ~~connected~~ relation cores ~~structures~~ TFAE

$\downarrow$ ① Forb($\mathcal{F}$) = CSP(H) for some H

② $\mathcal{F}$ is a finite set of relational trees

Forb() = Forb()

Homomorphism Order $\leq, < \rightsquigarrow A \to B$

$A < B$ is a gap if there is no C s.t. $A < C < B$.

(F, D), dual pair $\text{Forb}(\{F\}) = \text{CSP}(D)$

connected gap if B is connected

Thm| There is a 1-1 correspondence between connected gaps and dual pairs.

(Only gap in graphs is $\circ <$ 

 $\not\to = \to \circ$)

Pf| ① $A \overset{\text{gap}}{<} B$ and B connected. Let C be arb. structure

$A \leq A + (B \times C) \leq B$

($+$: disjoint union; $\times$: category product)

Two possibilities

① $B \times C \to A \Leftrightarrow C \to A^B$

② $B \to C$

we proved (B, A^B) is a dual pair

(Example| A, B graphs

$V(A^B) = \{ f : V(B) \to V(A) \}$

$(f, g) \in E(A^B) \Leftrightarrow \forall (u, v) \in E(B)$ holds $(f(u), f(v)) \in E(A)$)

② (F, D) dual pair $F \not\to A \Leftrightarrow A \to D$

Consider $A = D \times F$, $B = F$

then $A \leq B$

Suppose $A \leq C \leq B$

Suppose $B \not\to C \equiv F \not\to C$ so $C \to D, F$

So $C \to A$.

$\square$

Fork

G orient

Giv

Thm

G ∈

$\mathrm{Forb}(\underset{\text{finite}}{\mathcal{F}'}) = \{A' \mid F' \not\to A' \text{ for } F' \in \mathcal{F}'\}$

$\underset{\text{oriented}}{G} = (V, E) \rightsquigarrow G' = (V, E, E_1, E_2)$

Given G $\exists$ $G' \in \mathrm{Forb}(\mathcal{F}')$ $(*)$

Thm) (Kim, N.) Every NP problem is P-equivalent to a problem of type $(*)$.

$G \in \mathrm{CSP}(\triangle)$ $\quad G = (V, E) \rightsquigarrow G' = (V, E, \underbrace{U_1, U_2, U_3}_{\text{colors}})$

(3 forbidden relations $u_i - u_i$)