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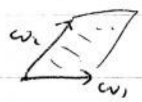
K3, Enriques surfaces.

§0. Introduction

elliptic curves. $E = \mathbb{C}/\Lambda$ $\quad \Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$



$H_1(E, \mathbb{Z}) \cong \mathbb{Z}^2$

$\quad \Psi$

γ_1, γ_2 a basis

$\exists_1 \omega_E$: a hol 1-form on E. unique up to const. $\quad \tau(E) := \int_{\gamma_1} \omega_E \Big/ \int_{\gamma_2} \omega_E$ $\quad$ FACT : $\operatorname{Im} \tau_E \neq 0$.

By changing γ_1 and γ_2, we may assume : $\operatorname{Im} \tau(E) > 0$. , $\tau(E) \in H^+ = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$.

γ_1', γ_2' : a basis $\quad \gamma_1' = a\gamma_1 + b\gamma_2$

$\quad \gamma_2' = c\gamma_1 + d\gamma_2$.

$$\int_{\gamma_1'} \omega_E \Big/ \int_{\gamma_2'} \omega_E = a \int_{\gamma_1} \omega_E + b \int_{\gamma_2} \omega_E \Big/ c \int_{\gamma_1} \omega_E + d \int_{\gamma_2} \omega_E = \frac{a\tau(E) + b}{c\tau(E) + d}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

$$\{\text{elliptic curves}\}/\cong \xrightarrow{\tau} H^+/SL(2, \mathbb{Z})$$

In case of K3. $\quad \omega_X$: a hol. 2-form. unique up to const

$$H^2(X, \mathbb{C}) \ni \omega_X : H_2(X, \mathbb{Z}) \to \mathbb{C} \qquad \begin{cases} \langle \omega_X, \omega_X \rangle = 0 \\ \langle \omega_X, \bar{\omega}_X \rangle > 0. \end{cases}$$
$$\gamma \longmapsto \int_\gamma \omega_X$$

K3 $\underset{\text{Torelli}}{=\!=}$ lattices

§1. Lattices.

§2. Periods of K3, Enriques surfaces

§3. Automorphisms of " $\quad$ "A Description of Aut(X) for a K3 X", $\quad$ " A relation with the Mathieu gr M_{23}.

§4. (Borcherds product ~~or pseudo~~
(automorphic forms on type IV domain with known zeros & poles

§1. Lattices.

$L \cong \mathbb{Z}^r$, $\quad \langle\, ,\, \rangle : L \times L \to \mathbb{Z}$ $\quad$ a non-deg. symm. bilinear form

$L \otimes \mathbb{R}$ $\quad$ sign $L = (p, q)$ $\qquad p = 0$ or $q = 0 \Rightarrow L$ is called definite, otherwise indefinite $(p > 0, q > 0.)$

$L^* := \operatorname{Hom}(L, \mathbb{Z}) \hookleftarrow L$ $\qquad A_L := L^*/L$ a finite abelian gr.

$\langle x, \cdot \rangle \longleftarrow\!\shortmid\; x$

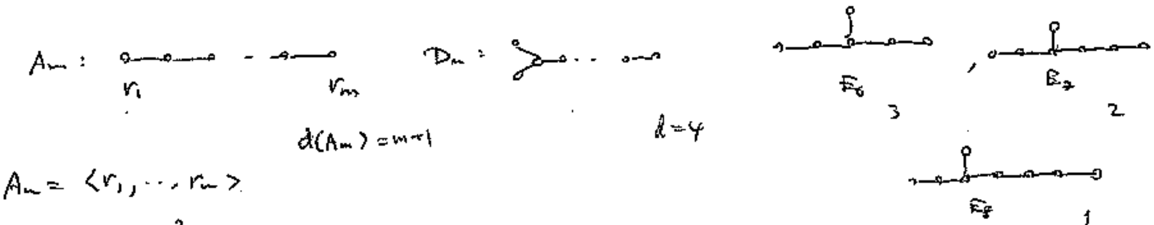
$d(L) = |A_L|$. $\qquad$ even, $\Leftrightarrow$ $\langle x, x \rangle \equiv 0 \ (2)$

L : unimodular if $L \xrightarrow{\sim} L^*$.

Example. $\quad U = (\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$, $\quad L(m) = (L, m\langle\, ,\, \rangle)$. $\quad U(2) = (\mathbb{Z}^2, \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix})$.

(neg.) def. lattice gen. by (−2)-vectors.
irred. root lattice

· · root lattices: A_n, D_n, E_k $(k=6,7,8)$
$(n\geq 1)$ $(n\geq 4)$



$d(A_m) = m+1$ $\qquad d=4$

$A_n = \langle r_1, \dots, r_n \rangle$

$\langle r_i, r_i \rangle = r_i^2 = -2$. $\qquad \langle r_i, r_j \rangle$ - line.

$$\overset{r_i}{\circ}\!\!-\!\!-\!\!\overset{r_j}{\circ}$$

· unimodular lattice

Prop. L: even, unimodular, $\mathrm{sign}=(p,q) \Rightarrow p-q \equiv 0 \ (8)$.

Prop. L: even, unimod., indef. ~~, sign(p,q)~~ $\Rightarrow$ L is uniquely determined by its $\mathrm{sign}(p,q)$

$$L \cong \begin{cases} U^q \oplus E_8(-1)^{\frac{p-q}{8}} & \text{if } p \geq q \\ U^p \oplus E_8^{\,q-p/8} & \text{if } q \geq p. \end{cases}$$

Remark definite case, not unique.

rk	8	16	24	32
#isom. class	1	2	24	80 million
	E_8	E_8^2, D_{16}^+	Niemeier lattice	

Coxeter # $= \dfrac{\#\text{roots}}{\mathrm{rank}(R)} = \dfrac{\#\text{roots}}{24}$

R	#roots	h
A_n	$n(n+1)$	$n+1$
D_n	$2n(n-1)$	$2(n-1)$
E_6	72	12
E_7	126	18
E_8	240	30.

N: a Niemeier lattice.
$\cup$
$R(N) :=$ root lattice $\Rightarrow$ $R(N) = \emptyset$ — Leech lattice
or
$\mathrm{rk}\, R(N) = 24$ — 23 isom. classes determined by $R(N)$.

Irred. comp. of $R(N)$ has the same Coxeter # h.

~~Example~~ $R(N) = A_1^{24}, A_2^{12}, \dots,$

discriminant quadratic form

L: even lattice

$$q_L : A_L = L^*/L \longrightarrow \mathbb{Q}/2\mathbb{Z}$$
$$x + L \longmapsto \langle x, x \rangle + 2\mathbb{Z}$$
$$b_L : A_L \times A_L \longrightarrow \mathbb{Q}/\mathbb{Z}$$

$q_L(x+y) = q_L(x) + q_L(y) + 2b_L(x,y)$ mod $2\mathbb{Z}$.

Overlattice $\quad L$: even lattice.

$A_L \supset H$ isotropic subgr. w.r.t. q_L $(q_L|H \equiv 0)$.

$L^* \supset L_H := \{ x \mid x \bmod L \in H \} \supset L$.
even lattice.

Originally,

$L' \supset L$ $\quad rk(L') = rk(L)$ $\Rightarrow$ L' is called an overlattice of L.
even lattice

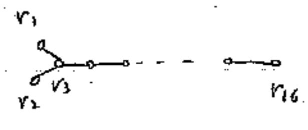
$$\{L' \mid L' \text{ overlattice of } L\} \xleftrightarrow{1:1} \{H < A_L \mid q_L|H \equiv 0\}$$

$$L' \longmapsto L'/L$$

$$L_H \longleftarrow H$$

Examples

① $L = D_{16}$. $\quad A_{D_{16}}$



$$= \left\langle \frac{r_1+r_3+\cdots+r_{16}}{2}, \frac{r_2+r_4+\cdots+r_{16}}{2} \right\rangle \cong A_{U(2)}$$

$$\frac{-2\times 8}{4}$$

$d(D_{16})=4$ $\quad D_{16} \underset{2}{\subset} \exists\, \Gamma_{16}$ $\quad d(\Gamma_{16})=1$.

② $L = A_1^{24}$ $\quad A_L \cong \mathbb{F}_2^{24} \ni (x_1,\dots,x_{24}) = x$ $\qquad x^2 = \sum -\frac{1}{2}x_i^2$.

$\cup$

$\exists\, \mathcal{G}$ a binary Golay code. $\qquad -\frac{1}{2}$

$\mathcal{G} \cong \mathbb{F}_2^{12}$ $\quad \cdot\ (1,\dots,1) \in \mathcal{G}$

$\Psi\ x$ $\quad \cdot$ # non-zero entries of x is a multiple by 4.

$\cdot$ # non-zero of $x \geq 8$, if $x \neq 0$

$d = 2^{24}$ $\quad \exists$

$L \subset N$ overlattice

2^{12}. $\qquad d(L) = [N:L]^2 d(N) \Rightarrow d(N) = 1$

$O(N)/W(N) \subset \mathfrak{S}_{24}$ $\quad \cancel{O(N) \cong W(A_1^{24}) \cdot \mathfrak{S}_{24}}$ $\qquad M_{24} = \{\sigma \in \mathfrak{S}_{24} \mid \sigma(\mathcal{G}) \subset \mathcal{G}\}$

Mathieu group
(sporadic finite simple gp of order $2^{10}\cdot 3^3\cdot 5\cdot 7\cdot 11\cdot 23$)
M_{23}: $2^7\cdot 3^2\cdot 5\cdot 7\cdot 11\cdot 23$

· primitive embeddings of even lattices into unimodular lattices

L : even unimodular

$\cup$ S a sublattice primitive (L/S : torsion free).

$T := S^\perp$ in L. primitive sublattice.

$S \oplus T \subset L$. an overlattice corr. to $\quad H := L/S\oplus T \subset A_{S\oplus T} = A_S \oplus A_T$

w.r.t. $q_S \oplus q_T$.

④

Prop

$$\begin{tikzcd} & A_S \oplus A_T \\ & H \arrow[u, phantom, "\cup"] \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\ A_S \arrow[rr, "\gamma = p_T \circ p_S^{-1}"'] & & A_T \end{tikzcd}$$

($pr_S : A_S \oplus A_T \to A_S$, $pr_T : A_S \oplus A_T \to A_T$)

$$q_S(x) + q_T(\gamma(x)) \equiv 0 \ (2) \quad (x \in A_S)$$

☺ $L \ni x = x_S + x_T$, $x_S \in S^*$, $x_T \in T^*$.

$\overline{x_S} = 0$ in A_S $\Rightarrow$ $x_T = x \in L$. T: prime $\Rightarrow$ $\overline{x_T} = 0$

$\Rightarrow$ $p_S : H \to A_S$ injective

$|H|^2 = |A_S| \cdot |A_T|$.

$\Rightarrow$ $p_S : H \xrightarrow{\cong} A_S$

Conversely

~~Cor.~~ **Prop** S, T: even lattices, $\gamma : A_S \xrightarrow{\cong} A_T$ isom. s.t. $q_T(\gamma(x)) = -q_S(x)$.

$\Rightarrow$ $\exists L$: even unimodular s.t. $S \overset{\exists}{\hookrightarrow} L$ primitive, $T = S^{\perp}$ in L.

☺ $H := \{ (x, \gamma(x)) \mid x \in A_S \} \subset A_S \oplus A_T = A_{S \oplus T}$. Isot. w.r.t. $q_S \oplus q_T$.

γ: isom $\Rightarrow$ $S \hookrightarrow L$ primitive

Example. $L = E_8$

S	A_1	A_2	A_3	A_4	D_4
T	E_7	E_6	D_5	A_4	D_4

Cor. L: even, unim., $S \subset L$ primitive, $T = S^{\perp}$, $g \in O(S)$, $g|A_S = id$.

$\Rightarrow$ $\exists \tilde{g} \in O(L)$ s.t. $\tilde{g}|S = g$, $\tilde{g}|T = id_T$

☺ $\tilde{g} = (g, 1_T) \in O(S \oplus T) \subset O(S^* \oplus T^*)$

$\tilde{g} | A_T \oplus A_S = id.$ $\Rightarrow$ $\tilde{g}(L) \subset L$.

Cor. $O(T) \twoheadrightarrow O(q_T)$.

$\Rightarrow$ $O(S) \ni {}^{\forall}g$. can be extended to $\exists \tilde{g} \in O(L)$.

for ${}^{\forall}p \neq 2$ $\left\{ \begin{array}{l} \cdot\ rk(T) \geq \ell(A_{T_p}) + 2 \\ \cdot\ rk\,T = \ell(A_{T_2}) \\ \Rightarrow q_{T_2} = u_2 \oplus q' \text{ or } v_2 \oplus q' \end{array} \right.$ $\Rightarrow$ T is unique & $O(T) \twoheadrightarrow O(q_T)$

T: even indef. lattice ~~(...)~~ s.t. $rk(T) \geq \ell(A_T) + 2$

Th (Nikulin) ~~T: even & indefinite of sgn (t_+, t_-).~~ ~~$q = q_T$,~~ ~~(t_+, t_-): integers, q: a discriminant $A \to \mathbb{Q}/2\mathbb{Z}$~~

~~Assume: $rk(T) \geq \ell(A_T) + 2$.~~ ~~$\exists!\, T$: even lattice of sgn (t_+, t_-), $q_T \cong q$.~~

$\Rightarrow$ ~~T is uniquely determined~~ unique ~~if $t_+ - t_- = \mathrm{sign}(q)$ (8), $t_+ > 0$, $t_- > 0$, $t_+ + t_- = rk(T) \geq 2 + \ell(A_q)$~~

§ Periods for K3, Enriques

X : a K3 $H^2(X,\mathbb{Z})$: even, unimodular lattice of sign $=(3,19)$

L : an abstract lattice

- $H^2(X,\mathbb{Z}) \xrightarrow[\cong]{\alpha_X} L$ $\alpha_X(\omega_X) \in L\otimes\mathbb{C}$

$$\begin{array}{ccc} H^2(X,\mathbb{Z}) & \xrightarrow{\alpha_X} & L \\ \cap & & \cap \\ H^2(X,\mathbb{C}) & \xrightarrow{\alpha_X} & L\otimes\mathbb{C} \end{array}$$

$\Omega := \{ [\omega] \in \mathbb{P}(L\otimes\mathbb{C}) \mid \langle\omega,\omega\rangle = 0,\ \langle\omega,\bar\omega\rangle > 0 \}$

$\uparrow$ the period domain of (X,α_X).

- $h \in L$ primitive $h^2 = 2d > 0$

$\langle h \rangle \hookrightarrow L$ primitive $h^\perp =: L_{2d}$ unique $\exists\ H^2(X,\mathbb{Z}) \xrightarrow{\cong} L$, $h_X \longrightarrow h$

$D_{2d} := \{ [\omega] \in \mathbb{P}(L_{2d}\otimes\mathbb{C}) \mid \quad \text{''} \quad \}$.

$\uparrow$ a bdd sym. domain of type IV.

$O(L_{2d})$ properly discont.

$\cup$

$\Gamma_{2d} := \{ \gamma \mid \gamma^* | A_{L_{2d}} = id \}$. D_{2d}/Γ_{2d}

- Enriques surfaces

$X \xrightarrow{\pi} Y$: an Enriques $\cong X/\langle\sigma\rangle$ X a K3. σ : fixed pt free invol.

$p_g = q = 0$ $2K_Y \sim 0$ $H^2(Y,\mathbb{Z})/\text{tors} \cong \mathrm{Pic}\,Y/\text{tors} \cong U \oplus E_8$

$L_X^{\pm} := \{ x \in H^2(X,\mathbb{Z}) \mid \sigma^* x = \pm x \}$. $q_{L^+} \cong q_{L^-}$.

$\Rightarrow$ $L_X^+ = \pi^* H^2(Y,\mathbb{Z}) \cong U(2) \oplus E_8(2)$ $L^- \cong U(2) \oplus U \oplus E_8(2)$

$O(L^-) \twoheadrightarrow O(q_{L^-})$.

$\iota : L \to L$ invol.

$L^{\pm} := \{ x \in L \mid \iota^* x = \pm x \} = \begin{cases} U(2) \oplus E_8(2) \\ U \oplus U(2) \oplus E_8(2) \end{cases}$

$D = D(\iota) = \{ [\omega] \in \mathbb{P}(L_- \otimes \mathbb{C}) \mid \langle\omega,\omega\rangle = 0,\ \langle\omega,\bar\omega\rangle > 0 \}$.   10-dim. domain of type IV.

$\circlearrowleft\ \Gamma = O(L_-)$.

Remark $\omega \in D$ $\exists X$: a K3 with $\alpha_X : H^2(X,\mathbb{Z}) \to L$, $\alpha_X(\omega_X) = \omega$.

However it may happen that X is not a cover of Enriques.

Assume : $\exists r \in L_-$ $r^2 = -2$, $\langle \omega, r \rangle = 0$. $S_X := \omega_X^\perp \cap H^2(X,\mathbb{Z})$ Picard lattice.

$\Rightarrow \exists \delta \in S_X$ $\delta^2 = -2$, $\iota^*(\delta) = -\delta$.

R.R. $\Rightarrow$ δ or $-\delta$ is effective.

If $\underbrace{\iota \text{ is}}_{\text{rep. by}}$ an automorphism, this does not happen.

$$\Big(D \setminus \bigcup_{\substack{r \in L_- \\ r^2 = -2}} r^\perp \Big) / \Gamma = (D/\Gamma) \setminus \mathcal{H}.$$

↑ irred. divisor

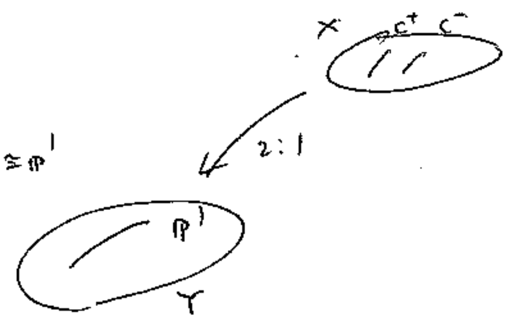
~~Boundary points~~

Remark a gen. Enriques does not contain $\mathbb{P}^1$.

gen. : $\rho(X) = 10$. i.e. $\mathrm{Pic}\, X = L_+$.

(?)

$Y \supset C \cong \mathbb{P}^1$



$(C^+ - C^-)^2 = -4$.

$C^+ - C^- \in L_-$.

Root invariants

$$\Delta_\pm := \Big\{ \delta_\pm \in S_X^\pm \;\Big|\; \delta_\pm^2 = -4,\ \exists \delta_\mp \in S_X^\mp \text{ s.t. } \delta_\mp^2 = -4,\ \frac{(\delta_\pm + \delta_\mp)}{2} \in S_X \Big\}.$$

$K := [\Delta_-](\tfrac{1}{2})$ a root lattice.

Nikulin root inv.

$$\xi : K/2K \longrightarrow S_X^+(\tfrac{1}{2}) \bmod 2.$$

$$\delta_- \longmapsto \delta_+$$

$(K, \mathrm{Ker}\,\xi)$
root invariant defined by Nikulin

Ex. (Hessian quartic)

$$F = \left(\sum_{i=1}^{5} \lambda_i x_i^3 = 0,\ \sum_{i=1}^{5} x_i = 0\right) \subset \mathbb{P}^4.$$ Sylvester form $(\lambda_i \neq 0)$

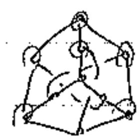
$$H = \left(\sum \frac{1}{\lambda_i x_i} = 0,\ \sum x_i = 0\right) \longleftarrow X \quad \text{a K3.}$$

Ψ 10 node $P_{ijk}: x_i = x_j = x_k = 0$ $\qquad \cup\ E_{ijk},\ L_{ij}$ $\quad (10)_3$-conf.

10 line $\ell_{ij}: x_i = x_j = 0$.

$(x_1 : \cdots : x_5) \to \left(\frac{1}{\lambda_1 x_1} : \cdots : \frac{1}{\lambda_5 x_5}\right)$ induces a fixed pt. free invol. σ of X.

$\sigma: L_{ij} \leftrightarrow E_{klm}$



Petersen graph $\mathfrak{S}_5$.

$\lambda_1 = \cdots = \lambda_5 = 1 \Rightarrow \mathrm{Aut}(X) \cong \mathfrak{S}_5$
(Kondo)

$E_6(2)$. $\quad K = E_6,\ K_a\zeta = 0$

$T \cong U \oplus U(2) \oplus A_2(2)$.

Mukai, Ohashi,

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§. Automorphism of K3, Enriques

X : alg. K3

$S_X = \mathrm{Pic}\, X$.

$\Delta(X) := \{\delta \in S_X \mid \delta^2 = -2\} \ni \delta \qquad s_\delta : x \to x + \langle x, \delta\rangle \delta \quad \in O(S_X)$

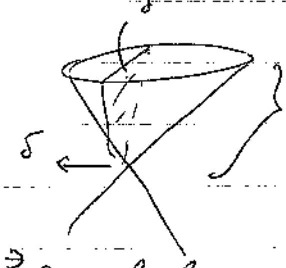
$\cup$ $\quad W(X) = \langle \{ s_\delta \mid \delta \in \Delta(X)\} \rangle$

$\delta^\perp$

$P(X) = \{x \in S_X \otimes \mathbb{R} \mid x^2 > 0\}$

$\cup$

$W(X) \curvearrowright P^+(X)$ a conn. comp. $\ni$ an ample class



δ $\quad P^+(X)$

$C(X) :=$ the conn. comp. of $P^+(X) \setminus \bigcup_{\delta \in \Delta(X)} \delta^\perp$ $\ni$ an ample class

a fund. dom. of $W(X)$

$\mathrm{Aut}(X) \longrightarrow \mathrm{Aut}(C(X)) := \{ g \in O(S_X) \mid g(C(X)) \subset C(X)\} \cong O(S_X)/\{\pm 1\}\cdot W(X)$

$\uparrow$ a finite kernel & cokernel

$\mathrm{Aut}(C(X)) \supset \mathrm{Ker}\{\mathrm{Aut}(C(X)) \longrightarrow O(q_{S_X})\}$ $\qquad \exists\, \tilde{\varphi} \in O(L)$ s.t. $\tilde{\varphi}|_{S_X} = \varphi,\ \tilde{\varphi}|T_X = id.$

a finite index

Cor. $|\mathrm{Aut}(X)| < \infty \iff [O(S_X) : W(X)] < \infty$ (Nikulin, Vinberg)

reflective $\qquad$ Esselmann $\qquad$ Vinberg the most alg. K3 $\qquad$ Kummer

??

G : a finite gr. $\curvearrowright X$ of K3.

Ψ

g. $\quad g^*\omega_X = \alpha(g)\cdot\omega_X \qquad \alpha(g)$: a root of 1.

$$G \xrightarrow{\alpha} \mathbb{C}^*$$
$$g \longmapsto \alpha_g$$

$$1 \to \operatorname{Ker}\alpha \to G \to \mathbb{Z}/n\mathbb{Z} \to 1$$

$\operatorname{Ker}\alpha$: symplectic; $\mathbb{Z}/n\mathbb{Z}$: non-symplectic

- $g^*\omega_X = \zeta_m \cdot \omega_X$.

$g^* | T_X \otimes \mathbb{Q} = \bigoplus$ irred. rep. of $\mathbb{Z}/m\mathbb{Z}$ of degree $\varphi(m)$

In particular $\varphi(m) \mid \operatorname{rk} T_X$.

$\operatorname{rk} T_X \leq 21$.

$m \leq 66$

$y^2 = x^3 + t^{12} - t$.

g : a finite symplect. auto. of K3

$\Rightarrow \quad |g| \leq 8$

$\lvert g\rvert$	2	3	4	5	6	7	8
# fixed pts	8	6	4	4	2	3	2.

$\lvert\sigma\rvert$	$(2)^8$	$(3)^6$	$(4)^4(2)^2$	$(5)^4$	$(6)^2(3)^2(2)^2$	$(7)^3$	$(8)^2(4)(2)$	$(11)^2$	$(14)(7)(2)$
									$(15)(5)(3)$
									(23)

$M_{23} \subset M_{24} \curvearrowright \Omega = \{1, 2, \cdots, 24\}$

Ψ $\qquad |\sigma| = |g|$

σ $\qquad$ # fixed pts of σ on Ω = # fixed pts of g on X.

$\because \quad g^*(x) = x \quad x \in T_X \otimes \mathbb{Q}$.

$\langle x, \omega_X \rangle = \langle g^*x, g^*\omega_X \rangle$

$= \langle x, \zeta_m \omega_X \rangle$

$\neq 1 \qquad x \in \omega_X^\perp = S_X$

G : a finite gr. of sympl. auto. of K3. $\Rightarrow G \hookrightarrow M_{23}$ with at least 5 orbits on Ω.

$H^*(X, \mathbb{Q})$

$\operatorname{Re}\omega_X, \operatorname{Im}\omega_X, \; H^0, H^4$, Kähler class

(Mukai) $\qquad$ Kondo

Ohashi

Enriques surface

generic $\begin{cases} \cdot\; \operatorname{rk}(S_X) = 10. \\ \cdot\; \omega_X \notin \text{a proper linear subsp. of } L \otimes \mathbb{C} \text{ defined over } \mathbb{Q}(\zeta_m) \end{cases}$

$\varphi(m) \leq 12$.

$$\Rightarrow \quad \operatorname{Aut}(X, \sigma)/\langle\sigma\rangle \cong \operatorname{Ker}\{O(L^+) \to O(q_{L^+})\}/\pm 1. \quad \text{(Barth–Peters)}$$

$\quad \| $

$\operatorname{Aut}(Y)$

Nikulin, Kondo $\quad$ There are 7 types.
two of them are 1-dim
others are isolated.

§. Borcherds products

L : even lattice of sign $(2, b)$.

$A_L = L^*/L \ni \gamma \qquad e_\gamma \in \mathbb{C}[A_L]$

$SL_2(\mathbb{Z}) : \curvearrowright \mathbb{C}[A_L] \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \; S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad SL_2(\mathbb{Z}/N\mathbb{Z})$

$N(\gamma, \delta)$, $N\gamma^2/2 \in \mathbb{Z}$

$\forall \gamma, \delta \in L^*$

$$\begin{cases} \rho_L(T) e_\gamma = e^{2\pi\sqrt{-1}\, q_L(\gamma)} e_\gamma \\ \rho_L(S) e_\gamma = \dfrac{i^{b-1}}{\sqrt{|A_L|}} \displaystyle\sum_{\delta \in A_L} e^{-2\pi\sqrt{-1}\, b_L(\gamma,\delta)} e_\delta \end{cases}$$

$f : H^+ \longrightarrow \mathbb{C}[A_L]$ a hol. & mero. at cusps is called a modular form of wt. k & type ρ_L.

$\circlearrowleft SL(2,\mathbb{Z}) \qquad \circlearrowleft \rho_L \qquad F\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \rho_L\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot F(\tau)$.

Th (Borcherds) f: a modular form of wt. $1-b$ & ρ_L.

$$\sum_{\gamma \in A_L} f_\gamma(\tau)\, e_\gamma = \sum_\gamma \sum_{n \in \mathbb{Q}} \underbrace{c_\gamma(n)}_{\in \mathbb{Z} \text{ for } n \leq 0} e^{2\pi\sqrt{-1} n\tau}\, e_\gamma.$$

$\Rightarrow \exists\, \Psi$: a mero. auto. form of wt. $c_0(0)/2$. with known zeros & poles on $\mathcal{D}(L)$

$\widetilde{\mathcal{D}}(L) := \{ Z \in L \otimes \mathbb{C} \mid \langle Z, Z \rangle = 0, \; \langle Z, \bar{Z} \rangle > 0 \}$

$\downarrow$

$\mathcal{D}(L) := \{ Z \in \mathbb{P}(L \otimes \mathbb{C}) \mid \quad \text{"} \quad \}$

$\Psi : \widetilde{\mathcal{D}}(L) \longrightarrow \mathbb{C}$ ~~mero. fct~~ an $\underbrace{\text{(mero)}}$ auto. form of wt k if Ψ is a mero fct. on $\widetilde{\mathcal{D}}(L)$ s.t. Ψ is homog. of degree $-k$ $\Psi(cZ) = c^{-k}\Psi(Z)$

inv. under $\Gamma \subset O(L)$ finite index

Ex $L = II_{2,26}$ $A_L = 0$

$f = \frac{1}{\Delta(\tau)}$ $\Delta(\tau) = q \prod_{n>0} (1-q^n)$

$= q^{-1} + 24 + \cdots$

Ψ : hol. aut. form on $D(L)$. of wt. 12 with zero alg. (−2)-vector

L_{2d} pol. K3 lattice of degree 2d.

$\curvearrowright$

L

$E_7 \subset L$

$E_7^\perp \cong L_{2,19}$

wt. $= 12 + 63 = 75$

$-\frac{1}{2}$ $56+1 = 57$

$U \oplus U \oplus E_8 \oplus E_7 \oplus \langle -2 \rangle$

ψ

r

1 mult.

Ex $L = U \oplus U(2) \oplus E_8$. $L \cong L^*(2)$.

$L^*/M = \mathbb{Z}/2 \times \mathbb{Z}/2$. $\mathbb{Z}/2 = \{0, 1\}$

$\underbrace{00, 01, 10}_{\text{isotropic}}, \underbrace{11}_{\text{non-isot.}}$

$f_{00} = 8\eta(2\tau)^8 / \eta(\tau)^{16} = 8 + 128q + \cdots$

$f_{01} = f_{10} = -$ 〃 $= -8 - 128q + \cdots$

$f_{11} = 8\eta(2\tau)^8/\eta(\tau)^{16} + \eta(\tau/2)^8/\eta(\tau)^{16} = q^{-1/2} + 36 q^{1/2} + \cdots$

$\exists \Psi$; an hol. aut. form of wt. 4 with zero div. (−1)-vectors

On $D(L \oplus \mathbb{Q})$

$(\Psi) = \mathcal{H}$. Yoshikawa