LECTURES IN TRANSCENDENTAL ALGEBRAIC GEOMETRY: ALGEBRAIC CYCLES WITH A SPECIAL EMPHASIS ON CALABI-YAU VARIETIES

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ABSTRACT. These notes form an expanded version of some introductory lectures to be delivered at the Workshop on Arithmetic and Geometry of $K3$ surfaces and Calabi-Yau Threefolds, August 16-25, 2011, at the Fields Institute. After presenting a general overview, we begin with some rudimentary aspects of Hodge theory and algebraic cycles. We then introduce Deligne cohomology, as well as generalized cycles that are connected to higher K-theory, and associated regulators. Finally, we specialize to the Calabi-Yau situation, and explain some recent developments in the field.

CONTENTS

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0 Introduction

These lecture notes concern that part of Calabi-Yau geometry that involves algebraic cycles - typically built up from special subvarieties, such as rational points and rational curves. From these algebraic cycles, one forms various doubly indexed groups, called higher Chow groups, that mimic simplicial homology theory in algebraic topology. These Chow groups, come equipped with various maps whose target is a certain transcendental cohomology theory called Deligne cohomology.

More precisely, these maps are called regulators, from the higher cycle groups of S. Bloch, denoted by $\mathrm{CH}^k(X,m)$, of a projective algebraic manifold X, to Deligne cohomology, viz.:

$$\mathrm{cl}_{r,m} : \mathrm{CH}^r(X,m) \to H^{2r-m}_{\mathcal{D}}\big(X, \mathbb{A}(r)\big), \tag{1}$$

where $\mathbb{A} \subseteq \mathbb{R}$ is a subring, $\mathbb{A}(r) := \mathbb{A}(2\pi\sqrt{-1})^r$ is called the "Tate twist", and as we will indicate below, some striking evidence that these regulator maps become highly interesting in the case where X is Calabi-Yau. More specifically, we consider the following case scenarios below.

When $m = 0$, the objects of interest are the null homologous codimension 2 (= dimension 1) cycles $\mathrm{CH}^2_{\mathrm{hom}}(X) = \mathrm{CH}_{1,\mathrm{hom}}(X)$ on a projective threefold X, and where in this case, (1) becomes the Abel-Jacobi map:

$$\Phi_2 : \mathrm{CH}^2_{\mathrm{hom}}(X) \to J^2(X) = \frac{\{H^{3,0}(X) \oplus H^{2,1}(X)\}^\vee}{H_3(X,\mathbb{Z})}, \tag{2}$$

defined by a process of integration, $J^2(X)$ being the Griffiths' jacobian of X. One of the reasons for introducing the Abel-Jacobi map is to study the Griffiths group $\mathrm{Griff}^2(X)\otimes\mathbb{Q}$. If we put $\mathrm{CH}^r_{\mathrm{alg}}(X)$ to be codimension r cycles algebraically equivalent to zero, then the Griffiths group is given by $\mathrm{Griff}^r(X) := \mathrm{CH}^r_{\mathrm{hom}}(X)/\mathrm{CH}^r_{\mathrm{alg}}(X)$.

When $m = 1$, the object of interest is the group

$$\mathrm{CH}^2(X,1) = \frac{\left\{\sum_{j,\mathrm{cd}_X Z_j=1} \big(f_j, Z_j\big) \,\middle|\, \begin{matrix} f_j \in \mathbb{C}(Z_j)^\times \\ \sum_j \mathrm{div}(f_j) = 0 \end{matrix}\right\}}{\mathrm{Image}\big(\text{Tame symbol}\big)},$$

on a projective algebraic surface X. If we mod out by the subgroup of $\mathrm{CH}^2(X,1)$ where the f_j's $\in \mathbb{C}^\times$, then we arrive at the quotient group of indecomposables $\mathrm{CH}^2_{\mathrm{ind}}(X,1)$ which plays an analogous role to the Griffiths group above. Moreover if we assume that the torsion part of $H^3(X,\mathbb{Z})$ is zero, then in this case (1) becomes a map:

$$\underline{\mathrm{cl}}_{2,1} : \mathrm{CH}^2_{\mathrm{ind}}(X,1) \to \frac{\big[H^{2,0}(X) \oplus H^{1,1}_{\mathrm{tr}}(X)\big]^\vee}{H_2(X,\mathbb{Z})}, \tag{3}$$

where $H^{1,1}_{\mathrm{tr}}(X)$ is the transcendental part of $H^{1,1}(X)$, being the orthogonal complement to the subgroup of algebraic cocyles.

In the case $m = 2$, the objects of interest are the group of symbols:

$$\mathrm{CH}^2(X,2) = \left\{\xi := \prod_j \{f_j, g_j\} \,\middle|\, \begin{matrix} f_j, g_j \in \mathbb{C}(X)^\times \\ \sum_{j,p\in X} \big((-1)^{\nu_p(f_j)\nu_p(g_j)}\big(\frac{f^{\nu_p(g_j)}}{g^{\nu_p(f_j)}}\big)(p), p\big) = 0 \end{matrix}\right\},$$

[ν_p = order of vanishing at p], on a smooth projective curve X. If we mod out by the subgroup of symbols $\{f, g\}$ where $f, g \in \mathbb{C}^\times$, then the group of interest is the quotient group of indecomposables $\mathrm{CH}^2_{\mathrm{ind}}(X, 2)$. In this case (1) becomes the real regulator:

$$r_{2,2} : \mathrm{CH}^2_{\mathrm{ind}}(X, 2) \to H^1(X, \mathbb{R}). \tag{4}$$

A first point we wish to make is that if X is a smooth projective variety of dimension d, where $1 \leq n \leq 3$, then the maps and objects

- $r_{2,2}$ in (4) and $\mathrm{CH}^2_{\mathrm{ind}}(X, 2) \otimes \mathbb{Q}$ for $d = 1$,

- $\underline{\mathrm{cl}}_{2,1}$ in (3) and $\mathrm{CH}^2_{\mathrm{ind}}(X, 1) \otimes \mathbb{Q}$ for $d = 2$,

- Φ_2 in (2) and $\mathrm{Griff}^2(X) \otimes \mathbb{Q}$ for $d = 3$,

become especially interesting and generally nontrivial in the case where X is a Calabi-Yau variety; moreover, in a sense that will be specified later, these maps are essentially "trivial" when restricted to indecomposables, for X either of "lower or higher order" to its Calabi-Yau counterpart. The reason for this appears to be to the abundance of special types of subvarieties on the Calabi-Yau varieties.

Several recent developments in the context of algebraic cycles are included in these lecture notes. The intended target audience is advanced graduate students, post-doctoral fellows and non-specialists.

1. Notation

Throughout these notes, and unless otherwise specified, $X = X/\mathbb{C}$ is a projective algebraic manifold, of dimension d. A projective algebraic manifold is the same thing as a smooth complex projective variety. If $V \subseteq X$ is an irreducible subvariety of X, then $\mathbb{C}(V)$ is the rational function field of V, with multiplicative group $\mathbb{C}(V)^\times$.

2. Some Hodge theory

Some useful reference material for this section is [16] and [24].

Let E^k_X = $\mathbb{C}$-valued C^∞ k-forms on X. We have the decomposition:

$$E^k_X = \bigoplus_{p+q=k} E^{p,q}_X, \quad \overline{E^{p,q}_X} = E^{q,p}_X,$$

where $E^{p,q}_X$ are the C^∞ (p, q)-forms which in local holomorphic coordinates $z = (z_1, \ldots, z_n) \in X$, are of the form:

$$\sum_{|I|=p, |J|=q} f_{IJ} dz_I \wedge d\overline{z}_J, \quad I = 1 \leq i_1 < \cdots < i_p \leq n,$$

$$J = 1 \leq j_1 < \cdots < j_q \leq n$$

$$dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}, \quad d\overline{z}_J = d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}.$$

One has the differential $d : E^k_X \to E^{k+1}_X$, and we define

$$H^k_{\mathrm{DR}}(X, \mathbb{C}) = \frac{\ker d : E^k_X \to E^{k+1}_X}{dE^{k-1}_X}.$$

The operator d decomposes into $d = \partial + \overline{\partial}$, where $\partial : E_X^{p,q} \to E_X^{p+1,q}$ and $\overline{\partial} : E_X^{p,q} \to E_X^{p,q+1}$. Further $d^2 = 0 \Rightarrow \partial^2 = \overline{\partial}^2 = 0 = \partial\overline{\partial} + \overline{\partial}\partial$.

The above decomposition descends to the cohomological level, viz.,

Theorem 2.1 (Hodge decomposition)**.**

$$H^k_{\text{sing}}(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \simeq H^k_{\text{DR}}(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X),$$

where $H^{p,q}(X) =$ *d-closed (p,q)-forms (modulo coboundaries), and*

$$\overline{H^{p,q}(X)} = H^{q,p}(X).$$

Furthermore:

$$H^{p,q}(X) \simeq \frac{E^{p,q}_{X,d-\text{closed}}}{\partial\overline{\partial}E_X^{p-1,q-1}}.$$

Some more terminology: *Hodge filtration.* Put

$$F^k H^i(X, \mathbb{C}) = \bigoplus_{p \geq k} H^{p,i-p}(X).$$

Now recall $\dim X = d$.

Theorem 2.2 (Poincaré and Serre Duality)**.** *The following pairings induced by*

$$(w_1, w_2) \mapsto \int_X w_1 \wedge w_2,$$

are non-degenerate:

$$H^k_{\text{DR}}(X, \mathbb{C}) \times H^{2d-k}_{\text{DR}}(X, \mathbb{C}) \to \mathbb{C},$$

$$H^{p,q}(X) \times H^{d-p,d-q}(X) \to \mathbb{C}.$$

Therefore $H^k(X) \simeq H^{2d-k}(X)^\vee,\ H^{p,q}(X) \simeq H^{d-p,d-q}(X)^\vee$

Example 2.3.

$$\frac{H^i(X, \mathbb{C})}{F^r H^i(X, \mathbb{C})} \simeq F^{d-r+1} H^{2n-i}(X, \mathbb{C})^\vee.$$

For the next three sections, the reader is encouraged to consult the "Lectures on Algebraic Cycles" on my website: `http://www.math.ualberta.ca/Lewis_JD.html`

3. Algebraic cycles (classical)

Recall $X/\mathbb{C}$ smooth projective, $\dim X = d$. For $0 \leq r \leq d$, put $z^r(X)$ ($= z_{d-r}(X)$) $=$ free abelian group generated by subvarieties of codim r ($= \dim\ d - r$) in X.

Example 3.1. (i) $z^d(X) = z_0(X) = \{\sum_{j=1}^M n_j p_j \mid n_j \in \mathbb{Z},\ p_j \in X\}$.

(ii) $z^0(X) = z_d(X) = \mathbb{Z}\{X\} \simeq \mathbb{Z}$.

(iii) Let $X_1 := V(z_2^2 z_0 - z_1^3 - z_0 z_1^2) \subset \mathbb{P}^2$, and $X_2 := V(z_2^2 z_0 - z_1^3 - z_1 z_0^2) \subset \mathbb{P}^2$. Then $3X_1 - 5X_2 \in z^1(\mathbb{P}^2) = z_1(\mathbb{P}^2)$.

(iv) $\operatorname{codim}_X V = r-1$, $f \in \mathbb{C}(V)^\times$. $\operatorname{div}(f) := (f) := (f)_0 - (f)_\infty \in z^r(X)$ (principal divisor). [Note: $\operatorname{div}(f)$ is easy to define, by first passing to a normalization of V (using the properties of DVR), together with a proper push-forward.]

Divisors in (iv) generate a subgroup

$$z^r_{\mathrm{rat}}(X) \subset z^r(X). \text{ (rational equivalence)}$$

Definition 3.2.

$$\mathrm{CH}^r(X) := z^r(X)/z^r_{\mathrm{rat}}(X),$$

is called the r-th Chow group of X.

Remark 3.3. On can show that $\xi \in z^r_{\mathrm{rat}}(X) \Leftrightarrow \exists\ w \in z^r(\mathbb{P}^1 \times X)$, each component of the support $|w|$ flat over $\mathbb{P}^1$, such that $\xi = w[0] - w[\infty]$. [Here $w[t] := \mathrm{pr}_{2,*}\big(\langle \mathrm{pr}_1^*(t) \bullet w\rangle_{\mathbb{P}^1\times X}\big)$.] If one replaces $\mathbb{P}^1$ by any choice of smooth connected curve Γ (not fixed!) and $0,\ \infty$ by any 2 points $P,\ Q \in \Gamma$, then one obtains the subgroup $z^r_{\mathrm{alg}}(X) \subset z^r(X)$ of cycles that are algebraically equivalent to zero. There is a fundamental class map (described later) $z^r(X) \to H^{2r}(X,\mathbb{Z})$ whose kernel is denoted by $z^r_{\mathrm{hom}}(X)$. One has inclusions:

$$z^r_{\mathrm{rat}}(X) \subseteq z^r_{\mathrm{alg}}(X) \subseteq z^r_{\mathrm{hom}}(X) \subset z^r(X).$$

Definition 3.4. Put

(i) $\mathrm{CH}^r_{\mathrm{alg}}(X) := z^r_{\mathrm{alg}}(X)/z^r_{\mathrm{rat}}(X)$

(ii) $\mathrm{CH}^r_{\mathrm{hom}}(X) := z^r_{\mathrm{hom}}(X)/z^r_{\mathrm{rat}}(X)$

(iii) $\mathrm{Griff}^r(X) := z^r_{\mathrm{hom}}(X)_{/}z^r_{\mathrm{alg}}(X) = \mathrm{CH}^r_{\mathrm{hom}}(X)/\mathrm{CH}^r_{\mathrm{alg}}(X)$. (Griffiths group)

The Griffiths group is known to be trivial in the cases $r = 0,\ 1,\ d$.

4. Generalized cycles

The basic idea is this:

$$\mathrm{CH}^r(X) = \mathrm{Coker}\bigg(\bigoplus_{\mathrm{cd}_X V = r-1} \mathbb{C}(V)^\times \xrightarrow{\mathrm{div}} z^r(X)\bigg).$$

In the context of Minor K-theory, this is just

$$\underbrace{\bigg(\to \cdots \bigoplus_{\mathrm{cd}_X V = r-2} K_2^M(\mathbb{C}(V))\bigg) \xrightarrow{\text{Tame}}}_{\text{building a complex on the left}} \bigoplus_{\mathrm{cd}_X V = r-1} K_1^M(\mathbb{C}(V)) \xrightarrow{\mathrm{div}} \bigoplus_{\mathrm{cd}_X V = r} K_0^M(\mathbb{C}(V))$$

For a field $\mathbb{F}$, one has the Milnor K-groups $K_\bullet^M(\mathbb{F})$, where $K_0^M(\mathbb{F}) = \mathbb{Z}$, $K_1^M(\mathbb{F}) = \mathbb{F}^\times$ and

$$K_2^M(\mathbb{F}) = \bigg\{\text{Symbols } \{a,b\} \ \bigg|\ a,b \in \mathbb{F}^\times\bigg\}\bigg/\left\{\begin{matrix}\text{Steinberg relations}\\ \{a_1a_2,b\} = \{a_1,b\}\{a_2,b\}\\ \{a,b\} = \{b,a\}^{-1}\\ \{a,1-a\} = \{a,-a\} = 1\end{matrix}\right\}.$$

One has a Gersten-Milnor resolution of a sheaf of Milnor K-groups on X, which leads to a complex whose last three terms and corresponding homologies for $0 \leq m \leq 2$ are:

$$
\begin{array}{ccccc}
\bigoplus_{\mathrm{cd}_X Z=r-2} K_2^M(\mathbb{C}(Z)) & \xrightarrow{T} & \bigoplus_{\mathrm{cd}_X Z=r-1} \mathbb{C}(Z)^{\times} & \xrightarrow{\mathrm{div}} & \bigoplus_{\mathrm{cd}_X Z=r} \mathbb{Z} \\
\updownarrow & & \updownarrow & & \updownarrow \\
\mathrm{CH}^r(X,2) & & \mathrm{CH}^r(X,1) & & \mathrm{CH}^r(X,0)
\end{array}
\tag{5}
$$

where div is the divisor map of zeros minus poles of a rational function, and T is the Tame symbol map. The Tame symbol map

$$T : \bigoplus_{\mathrm{cd}_X Z=r-2} K_2^M(\mathbb{C}(Z)) \to \bigoplus_{\mathrm{cd}_X D=r-1} K_1^M(\mathbb{C}(D)),$$

is defined as follows. First $K_2^M(\mathbb{C}(Z))$ is generated by symbols $\{f, g\}$, $f,\ g \in \mathbb{C}(Z)^{\times}$.

For $f,\ g \in \mathbb{C}(Z)^{\times}$,

$$T\big(\{f, g\}\big) = \sum_D (-1)^{\nu_D(f)\nu_D(g)} \left(\frac{f^{\nu_D(g)}}{g^{\nu_D(f)}}\right)_D,$$

where $\big(\cdots\big)_D$ means restriction to the generic point of D, and ν_D represents order of a zero or pole along an irreducible divisor $D \subset Z$.

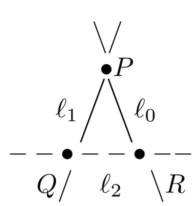
Example 4.1. Taking cohomology of the complex in (5), we have:

(i) $\mathrm{CH}^r(X) := \mathrm{CH}^r(X, 0) =$ free abelian group generated by subvarieties of codimension r in X, modulo divisors of rational functions on subvarieties of codimension $r - 1$ in X.

(ii) $\mathrm{CH}^r(X, 1)$ is represented by classes of the form $\xi = \sum_j (f_j, D_j)$, where $\mathrm{codim}_X D_j = r - 1$, $f_j \in \mathbb{C}(D_j)^{\times}$, and $\sum \mathrm{div}(f_j) = 0$ (and modulo the image of the Tame symbol).

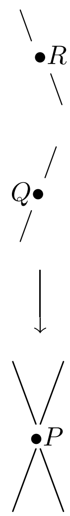
(iii) $\mathrm{CH}^r(X, 2)$ is represented by classes in the kernel of the Tame symbol, modulo the image of a higher Tame symbol.

Example 4.2. (i) $X = \mathbb{P}^2$, with homogeneous coordinates $[z_0, z_1, z_2]$. $\mathbb{P}^1 = \ell_j := V(z_j)$, $j = 0, 1, 2$. Let $P = [0, 0, 1] = \ell_0 \cap \ell_1$, $Q = [1, 0, 0] = \ell_1 \cap \ell_2$, $R = [0, 1, 0] = \ell_0 \cap \ell_2$. Introduce $f_j \in \mathbb{C}(\ell_j)^{\times}$, where $(f_0) = P - R$, $(f_1) = Q - P$, $(f_2) = R - Q$. Then $\xi := \sum_{j=0}^{2} (f_j, \ell_j) \in \mathrm{CH}^2(\mathbb{P}^2, 1)$ represents a higher Chow cycle.



Exercise 4.3. Show that $\xi \neq 0$. [Hint: Using a well-known fact that $\mathrm{CH}^2(\mathbb{P}^1, 1) \simeq \mathbb{C}^{\times}$, choose a suitable line $\mathbb{P}^1 \subset \mathbb{P}^2$ and show that $\xi\big|_{\mathbb{P}^1} \neq 1 \in \mathbb{C}^{\times}$.]

(ii) Again $X = \mathbb{P}^2$. Let $C \subset X$ be the nodal rational curve given by $z_2^2 z_0 = z_1^3 + z_0 z_1^2$ (In affine coordinates $(x, y) = (z_1/z_0, z_2/z_0) \in \mathbb{C}^2$, C is given by $y^2 = x^3 + x^2$). Let $\tilde{C} \simeq \mathbb{P}^1$ be the normalization of C, with morphism $\pi : \tilde{C} \to C$. Put $P = (0, 0) \in C$ (node) and let $R + Q = \pi^{-1}(P)$. Choose $f \in \mathbb{C}(\tilde{C})^\times = \mathbb{C}(C)^\times$, such that $(f)_{\tilde{C}} = R - Q$. Then $(f)_C = 0$ and hence $(f, C) \in \mathrm{CH}^2(\mathbb{P}^2, 1)$ defines a higher Chow cycle.



5. A detour via Milnor K-theory and the Gersten-Milnor complex

This section provides some of the foundations for the previous section. In the first part of this section, we follow closely the treatment of Milnor K-theory provided in [2], which provides the basic foundation for the definitions of higher Chow cycles. The reader with pressing obligations who prefers to work with concrete examples may skip this section, without losing sight of the main ideas presented in these notes.

Let $\mathbb{F}$ be a field, with multiplicative group $\mathbb{F}^\times$, and put $T(\mathbb{F}^\times) = \coprod_{n \geq 0} T^n(\mathbb{F}^\times)$, the tensor product of the $\mathbb{Z}$-module $\mathbb{F}^\times$. Here $T^0(\mathbb{F}^\times) := \mathbb{Z}$, $\mathbb{F}^\times = T^1(\mathbb{F}^\times)$, $a \mapsto [a]$. If $a \neq 0, 1$, set $r_a = [a] \otimes [1 - a] \in T^2(\mathbb{F}^\times)$. The two-sided ideal R generated by r_a is graded, and we put:

$$K_\bullet^M \mathbb{F} = \frac{T(\mathbb{F}^\times)}{R} = \coprod_{n \geq 0} K_n^M \mathbb{F}, \quad \text{(Milnor } K\text{−theory).}$$

For example, $K_0(\mathbb{F}) = \mathbb{Z}$, $K_1(\mathbb{F}) = \mathbb{F}^\times$, and $K_2^M(\mathbb{F})$ is the abelian group generated by symbols $\{a, b\}$, subject to the Steinberg relations:

$$\{a_1 a_2, b\} = \{a_1, b\}\{a_2, b\}$$
$$\{a, 1 - a\} = 1. \quad \text{for } a \neq 0, 1$$
$$\{a, b\} = \{b, a\}^{-1}$$
$$\{a, -a\} = 1.$$

Furthermore, one can also show that:

$$\{a, a\} = \{-1, a\} = \{a, a^{-1}\} = \{a^{-1}, a\}. \tag{2.1}$$

Quite generally, one can argue that $K_n^M(\mathbb{F})$ is generated $\{a_1, \ldots, a_n\}$,

$a_1, \ldots, a_n \in \mathbb{F}^\times$, subject to:

$$\text{(i)} \qquad (a_1, \ldots, a_n) \mapsto \{a_1, \ldots, a_n\},$$

is a multilinear function from $\mathbb{F}^\times \times \cdots \times \mathbb{F}^\times \to K_n^M(\mathbb{F})$,

$$\text{(ii)} \qquad \{a_1, \ldots, a_n\} = 0,$$

if $a_i + a_{i+1} = 1$ for some $i < n$.

Assume given a field $\mathbb{F}$ with discrete valuation $\nu : \mathbb{F}^\times \to \mathbb{Z}$, with corresponding discrete valuation ring $\{a \in \mathbb{F} \mid \nu(a) \geq 0\}$, and residue field $\mathbf{k}(\nu)$. One has maps $T : K_\bullet^M(\mathbb{F}) \to K_{\bullet-1}^M(\mathbf{k}(\nu))$. Choose $\pi \in \mathbb{F}^\times$ such that $\nu(\pi) = 1$. If we write $a = a_0\pi^i,\ b = b_0\pi^j \in K_1^M(\mathbb{F})$, then $T(a) = i \in \mathbb{Z} = K_0^M(\mathbf{k}(\nu))$ and

$$T\{a, b\} = (-1)^{ij}\,\frac{a^j}{b^i} \in \mathbf{k}(\nu)^\times = K_1^M(\mathbf{k}(\nu)) \quad \text{(Tame symbol)}.$$

5.1. **Sheafifying everything.** Let $\mathcal{O}_X$ be the sheaf of regular functions on X, with sheaf of units $\mathcal{O}_X^\times$. As in [21], we put

$$\mathcal{K}_{r,X}^M := \frac{\left(\mathcal{O}_X^\times \otimes \cdots \otimes \mathcal{O}_X^\times\right)}{\mathcal{J}}, \quad \text{(Milnor sheaf)},$$

where $\mathcal{J}$ is the subsheaf of the tensor product generated by sections of the form:

$$\left\{\tau_1 \otimes \cdots \otimes \tau_k \;\middle|\; \tau_i + \tau_j = 1, \quad \text{for some } i \text{ and } j,\ i \neq j\right\}.$$

[For example, $\mathcal{K}_{1,X}^M = \mathcal{O}_X^\times$.]

Introduce the Gersten-Milnor complex (a flasque resolution of $\mathcal{K}_{r,X}^M$, see [15], [22]):

$$\mathcal{K}_{r,X}^M \to K_r^M(\mathbb{C}(X)) \to \bigoplus_{\mathrm{cd}_X Z = 1} K_{k-1}^M(\mathbb{C}(Z)) \to \cdots$$

$$\to \bigoplus_{\mathrm{cd}_X Z = r-2} K_2^M(\mathbb{C}(Z)) \to \bigoplus_{\mathrm{cd}_X Z = r-1} K_1^M(\mathbb{C}(Z)) \to \bigoplus_{\mathrm{cd}_X Z = r} K_0^M(\mathbb{C}(Z)) \to 0.$$

We have

$$K_0^M(\mathbb{C}(Z)) = \mathbb{Z}, \qquad K_1^M(\mathbb{C}(Z)) = \mathbb{C}(Z)^\times,$$
$$K_2^M(\mathbb{C}(Z)) = \left\{\text{symbols } \{f, g\} \;\big/\; \text{Steinberg relations}\right\}.$$

The last three terms of this complex then are:

$$\bigoplus_{\mathrm{cd}_X Z = r-2} K_2^M(\mathbb{C}(Z)) \xrightarrow{T} \bigoplus_{\mathrm{cd}_X Z = r-1} \mathbb{C}(Z)^\times \xrightarrow{\mathrm{div}} \bigoplus_{\mathrm{cd}_X Z = r} \mathbb{Z} \to 0$$

where div is the divisor map of zeros minus poles of a rational function, and T is the Tame symbol map

$$T : \bigoplus_{\mathrm{codim}_X Z = r-2} K_2^M(\mathbb{C}(Z)) \to \bigoplus_{\mathrm{codim}_X D = r-1} K_1^M(\mathbb{C}(D)),$$

defined earlier.

Definition 5.2. For $0 \leq m \leq 2$,

$$\mathrm{CH}^r(X, m) = H_{\mathrm{Zar}}^{r-m}(X, \mathcal{K}_{r,X}^M).$$

Remark 5.3. The higher Chow groups $\mathrm{CH}^r(W, m)$ are defined for any non-negative r & m. See [3], and quasi-projective variety W over a field k.

6. Hypercohomology

An excellent reference for this is the chapter on spectral sequences in [16].

The reader familiar with hypercohomology can obviously skip this section. Let $(\mathcal{S}^{\bullet\geq 0}, d)$ be a (bounded) complex of sheaves on X. One has a Cech double complex

$$\big(C^\bullet(\mathcal{U}, \mathcal{S}^\bullet),\ d,\ \delta\big),$$

where $\mathcal{U}$ is an open cover of X. The k-th hypercohomology is given by the k-th total cohomology of the associated single complex

$$\big(M^\bullet := \oplus_{i+j=\bullet} C^i(\mathcal{U}, \mathcal{S}^j),\ D = d \pm \delta\big),$$

viz.,

$$\mathbb{H}^k(\mathcal{S}^\bullet) := \lim_{\vec{\mathcal{U}}} H^k(M^\bullet).$$

Associated to the double complex are two filtered subcomplexes of the associated single complex, with two associated Grothendieck spectral sequences abutting to $\mathbb{H}^k(\mathcal{S}^\bullet)$ (where $p + q = k$):

$${}'E_2^{p,q} := H_\delta^p(X, \mathcal{H}_d^q(\mathcal{S}^\bullet))$$

$${}''E_2^{p,q} := H_d^p(H_\delta^q(X, \mathcal{S}^\bullet)$$

The first spectral sequence shows that quasi-isomorphic complexes yield the same hypercohomology:

Alternate take. Two complexes of sheaves $\mathcal{K}_1^\bullet$, $\mathcal{K}_2^\bullet$ are said to be quasi-isomorphic if there is a morphism $h : \mathcal{K}_1^\bullet \to \mathcal{K}_2^\bullet$ inducing an isomorphism on cohomology $h_* : \mathcal{H}^\bullet(\mathcal{K}_1^\bullet) \xrightarrow{\sim} \mathcal{H}^\bullet(\mathcal{K}_2^\bullet)$. Take a complex of acyclic sheaves $(\mathcal{K}^\bullet, d)$ (viz., $H^{i>0}(X, \mathcal{K}^j) = 0$ for all j) quasi-isomorphic to $\mathcal{S}^\bullet$. Then

$$\mathbb{H}^k(\mathcal{S}^\bullet) := H^i\big(\Gamma(\mathcal{K}^\bullet)\big).$$

For example if $\mathcal{L}^{\bullet,\bullet}$ is an [double complex] acyclic resolution of $\mathcal{S}^\bullet$, then the associated single complex $\mathcal{K}^\bullet = \oplus_{i+j=\bullet} \mathcal{L}^{i,j}$ is acyclic and quasi-isomorphic to $\mathcal{S}^\bullet$.

Example 6.1. Let $(\Omega_X^\bullet, d)$, $(\mathcal{E}_X^\bullet, d)$ be complexes of sheaves of holomorphic and $\mathbb{C}$-valued C^∞ forms respectively. By the holomorphic and C^∞ Poincaré lemmas, one has quasi-isomorphisms:

$$(\mathbb{C} \to 0 \to \cdots) \xrightarrow{\approx} (\Omega_X^\bullet, d) \xrightarrow{\approx} (\mathcal{E}_X^\bullet, d),$$

where the latter two are Hodge filtered. The first spectral sequence of hypercohomology shows that

$$H^k(X, \mathbb{C}) \simeq \mathbb{H}^k(\mathbb{C} \to 0 \to \cdots) \simeq \mathbb{H}^k((F^p)\Omega_X^\bullet) \simeq \mathbb{H}^k((F^p)\mathcal{E}_X^\bullet).$$

The second spectral sequence of hypercohomology applied to the latter term, using the known acyclicity of $\mathcal{E}_X^\bullet$, yields

$$\mathbb{H}^k(F^p\mathcal{E}_X^\bullet) \simeq \frac{\ker d : F^pE_X^k \to F^pE_X^k}{dF^pE_X^{k-1}} \simeq F^pH_{\mathrm{DR}}^k(X),$$

where the latter isomorphism is due to the Hodge to de Rham spectral sequence.

7. Deligne cohomology

A standard reference for this section is [14]. For a subring $\mathbb{A} \subseteq \mathbb{R}$, we introduce the Deligne complex

$$\mathbb{A}_{\mathcal{D}}(r): \quad \mathbb{A}(r) \to \underbrace{\mathcal{O}_X \to \Omega_X^1 \to \cdots \to \Omega_X^{r-1}}_{\text{call this } \Omega_X^{\bullet < r}}.$$

Definition 7.1. Deligne cohomology is given by the hypercohomology:

$$H_{\mathcal{D}}^i(X, \mathbb{A}(r)) = \mathbb{H}^i(\mathbb{A}_{\mathcal{D}}(r)).$$

From Hodge theory, one has the isomorphisms

$$\mathbb{H}^i(\Omega_X^{\bullet \geq r}) \simeq F^r H^i(X, \mathbb{C}), \quad \mathbb{H}^i(\Omega_X^{\bullet < r}) \simeq \frac{H^i(X, \mathbb{C})}{F^r H^i(X, \mathbb{C})}.$$

Thus applying $\mathbb{H}^\bullet(-)$ to the short exact sequence:

$$0 \to \Omega_X^{\bullet < r}[-1] \to \mathbb{A}_{\mathcal{D}}(r) \to \mathbb{A}(r) \to 0,$$

yields the short exact sequence:

$$\begin{aligned} 0 \to \frac{H^{i-1}(X, \mathbb{C})}{H^{i-1}(X, \mathbb{A}(r)) + F^r H^{i-1}(X, \mathbb{C})} &\to H_{\mathcal{D}}^i(X, \mathbb{A}(r)) \\ \to H^i(X, \mathbb{A}(r)) \cap F^r H^i(X, \mathbb{C}) \to 0. \end{aligned} \tag{6}$$

If we put $\mathbb{A} = \mathbb{Z}$, and $i = 2r$, then (6) becomes:

$$0 \to J^r(X) \to H_{\mathcal{D}}^i(X, \mathbb{Z}(r)) \to \mathrm{Hg}^r(X) \to 0,$$

where $J^r(X)$ is the Griffiths jacobian, and the Hodge group

$$H^{2r}(X, \mathbb{Z}(r)) \cap F^r H^{2r}(X, \mathbb{C})$$

is more precisely given by $\mathrm{Hg}^r(X) = \{w \in H^{2r}(X, \mathbb{Z}(r)) \mid w \in H^{r,r}(X, \mathbb{C})$, via the map $H^{2r}(X, \mathbb{Z}(r)) \to H^{2r}(X, \mathbb{C})$, induced by the inclusion $\mathbb{Z}(r) \hookrightarrow \mathbb{C}\}$.

Next, if $\mathbb{A} = \mathbb{Z}$ and $i \leq 2r - 1$, then from Hodge theory, $H^{2r-1}(X, \mathbb{Z}(r)) \cap F^r H^{2r-1}(X, \mathbb{C})$ is torsion. In particular, there is a short exact sequence:

$$0 \to \frac{H^{i-1}(X, \mathbb{C})}{F^r H^{i-1}(X, \mathbb{C}) + H^{i-1}(X, \mathbb{Z}(r))} \to H_{\mathcal{D}}^i(X, \mathbb{Z}(r)) \to H_{\mathrm{tor}}^i(X, \mathbb{Z}(r)) \to 0,$$

where $H_{\mathrm{tor}}^i(X, \mathbb{Z}(r))$ is the torsion subgroup of $H^i(X, \mathbb{Z}(k))$. The compatibility of Poincaré and Serre duality yields the isomorphism:

$$\frac{H^{i-1}(X, \mathbb{C})}{F^r H^{i-1}(X, \mathbb{C}) + H^{i-1}(X, \mathbb{Z}(r))} \simeq \frac{F^{d-r+1} H^{2d-i+1}(X, \mathbb{C})^\vee}{H_{2d-i+1}(X, \mathbb{Z}(d-r))}.$$

Finally, if $\mathbb{A} = \mathbb{R}$ and $i = 2r - 1$, then $H_{\mathrm{tor}}^i(X, \mathbb{R}(r)) = 0$; moreover if we set

$$\pi_{r-1} : \mathbb{C} = \mathbb{R}(r) \oplus \mathbb{R}(r-1) \to \mathbb{R}(r-1)$$

to be the projection, then we have the isomorphisms:

$$\begin{aligned} H^{2r-1}_{\mathcal{D}}(X,\mathbb{R}(r)) &\simeq \frac{H^{2r-1}(X,\mathbb{C})}{F^rH^{2r-1}(X,\mathbb{C})+H^{2r-1}(X,\mathbb{R}(r))} \\ &\underset{\simeq}{\xrightarrow{\pi_{r-1}}} H^{r-1,r-1}(X,\mathbb{R})\otimes\mathbb{R}(r-1) \\ &=: H^{r-1,r-1}(X,\mathbb{R}(r-1)) \\ &\simeq \Big\{H^{d-r+1,d-r+1}(X,\mathbb{R}(d-r+1))\Big\}^{\vee}. \end{aligned}$$

8. Examples of $H^{r-m}_{\mathrm{Zar}}(X,\mathcal{K}^M_{r,X})$ and corresponding regulators

8.1. **Case $m=0$ and CY threefolds.** In this case, one works with the commutative diagram:

$$\begin{array}{ccc} \bigoplus_{\mathrm{codim}_X Z=r-1} K_1^M(\mathbb{C}(Z)) & \to & \bigoplus_{\mathrm{codim}_X Z=r} K_0^M(\mathbb{C}(Z)) \\ \|\wr & & \|\wr \\ \bigoplus_{\mathrm{codim}_X Z=r-1} \mathbb{C}(Z)^\times & \overset{\mathrm{div}}{\to} & \bigoplus_{\mathrm{codim}_X Z=r}\mathbb{Z} \end{array}.$$

It easily follows from this that:

$$H^r_{\mathrm{Zar}}(X,\mathcal{K}^M_{r,X}) \simeq \frac{\Gamma\big(\bigoplus_{\mathrm{codim}_X Z=r}\mathbb{Z}\big)\ = z^r(X)}{\mathrm{div}\big(\Gamma(\bigoplus_{\mathrm{codim}_X Z=r-1}\mathbb{C}(Z)^\times)\big)}$$

$$= \left\{\frac{\text{Free abelian group of codim } r \text{ subvarieties in } X}{\text{Divisors of rational functions on } Z \text{ of codim } r-1 \text{ in } X}\right\} =: \mathrm{CH}^r(X),$$

i.e. gives the classical description of $\mathrm{CH}^r(X)$. The fundamental class map:

$$\mathrm{cl}_r : \mathrm{CH}^r(X) \to H^{2r}_{\mathrm{DR}}(X,\mathbb{C}) \simeq H^{2d-2r}_{\mathrm{DR}}(X,\mathbb{C})^\vee,$$

can be defined in a number of equivalent ways:

(i) The $d\log$ map $\mathcal{K}^M_{r,X}\to\Omega^r_X$, $\{f_1,...,f_r\}\mapsto\bigwedge_j d\log f_j$, induces a morphism of complexes $\{\mathcal{K}^M_{r,X}\to 0\}\to\Omega^{\bullet\geq r}_X[r]$, and thus

$$\begin{aligned} \mathrm{CH}^r(X) = H^r_{\mathrm{Zar}}(X,\mathcal{K}^M_{r,X}) &= \mathbb{H}^r\big(\{\mathcal{K}^M_{r,X}\to 0\}\big) \to \mathbb{H}^r\big(\Omega^{\bullet\geq r}_X[r]\big) \\ &= \mathbb{H}^{2r}(\Omega^{\bullet\geq r}_X) = F^rH^{2r}_{\mathrm{DR}}(X,\mathbb{C}). \end{aligned}$$

(ii) Let $V\subset X$ be a subvariety of codimension r in X, and $\{w\}\in H^{2d-2r}_{\mathrm{DR}}(X,\mathbb{C})$, (de Rham cohomology). Define $\mathrm{cl}_r(V)(w) = \frac{1}{(2\pi\sqrt{-1})^{d-r}}\int_{V^*}w$, and extend to $\mathrm{CH}^r(X)$ by linearity, where $V^*=V\backslash V_{\mathrm{sing}}$. [Note that $\dim_{\mathbb{R}}V=2d-2r$.] The easiest way to show that cl_r is well-defined (closed current, and integrally defined) is to first pass to a desingularization of V above, and apply some standard Stokes' theorem and fundamental class arguments.

(iii) Thirdly one has a fundamental class generator $\{V\}\in H_{2d-2r}(V,\mathbb{Z}(d-r))\simeq H^{2r}_V(X,\mathbb{Z}(r))\to H_{2d-2r}(X,\mathbb{Z}((d-r))\simeq H^{2r}(X,\mathbb{Z}(r))$.

In summary we have

$$\mathrm{cl}_r : \mathrm{CH}^r(X) \to \mathrm{Hg}^r(X) := H^{2r}(X, \mathbb{Z}(r)) \cap H^{r,r}(X).$$

This map fails to be surjective in general for $r > 1$ (cf. [24]).

Conjecture 8.2 (Hodge)**.**

$$\mathrm{cl}_r : \mathrm{CH}^r(X) \otimes \mathbb{Q} \to H^{2r}(X, \mathbb{Q}(r)) \cap H^{r,r}(X),$$

is surjective.

Next, the Abel-Jacobi map:

$$\Phi_r : \mathrm{CH}^r_{\mathrm{hom}}(X) \to J^r(X),$$

is defined as follows. Recall that

$$J^r(X) = \frac{H^{2r-1}(X, \mathbb{C})}{F^r H^{2r-1}(X, \mathbb{C}) + H^{2r-1}(X, \mathbb{Z}(r))} \simeq \frac{F^{d-r+1} H^{2d-2r+1}(X, \mathbb{C})^\vee}{H_{2d-2r+1}(X, \mathbb{Z}(d-r))},$$

is a compact complex torus, called the Griffiths jacobian.

Prescription for Φ_r: Let $\xi \in \mathrm{CH}^r_{\mathrm{hom}}(X)$. Then $\xi = \partial\zeta$ bounds a $2d - 2r + 1$ real dimensional chain ζ in X. Let $\{w\} \in F^{d-r+1} H^{2d-2r+1}(X, \mathbb{C})$. Define:

$$\Phi_r(\xi)(\{w\}) = \int_\zeta w \quad \text{(modulo periods)}.$$

That Φ_r is well-defined follows from the fact that $F^\ell H^i(X, \mathbb{C})$ depends only on the complex structure of X, namely

$$F^\ell H^i(X, \mathbb{C}) \simeq \frac{F^\ell E^i_{X, d-\mathrm{closed}}}{d\big(F^\ell E^{i-1}_X\big)},$$

where we recall that E^i_X are the C^∞ complex-valued i-forms on X.

Theorem 8.3. *If* $F^{r-1} H^{2r-1}(X, \mathbb{C}) \cap H^{2r-1}(X, \mathbb{Q}(r)) = 0$*, then there is an induced map*

$$\underline{\Phi}_r : \mathrm{Griff}^r(X) \to J^r(X).$$

In particular $\Phi_r\big(\mathrm{CH}^r_{\mathrm{alg}}(X)\big) = 0 \in J^r(X)$*. This is the case for a very general CY threefold with* $r = 2$*.*

Both maps (cl_r, Φ_r) can be combined to give

$$\mathrm{cl}_{r,0} : \mathrm{CH}^r(X) = \mathrm{CH}^r(X, 0) \to H^{2r}_{\mathcal{D}}(X, \mathbb{Z}(r)),$$

with commutative diagram:

$$\begin{array}{ccccccccc}
0 \to & \mathrm{CH}^r_{\mathrm{hom}}(X) & \to \mathrm{CH}^r(X) \to & \frac{\mathrm{CH}^r(X)}{\mathrm{CH}^r_{\mathrm{hom}}(X)} & \to 0 \\
 & \Phi_r \downarrow & \mathrm{cl}_{r,0} \downarrow & \mathrm{cl}_r \downarrow & \\
0 \to & J^r(X) & \to H^{2r}_{\mathcal{D}}(X, \mathbb{Z}(r)) \to & \mathrm{Hg}^r(X) & \to 0.
\end{array}$$

The map $\mathrm{cl}_{r,0}$ can be defined using a local version of an exact sequence similar to (6), using a cone complex description of $H^\bullet_{\mathcal{D}}(X, \mathbb{Z}(\bullet))$, together with a weak purity argument.

8.4. **Deligne cohomology and normal functions.** Suppose that $\xi \in \mathrm{CH}^r(X)$ is given and that $Y \subset X$ is a smooth hypersurface. Then there is a commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^r(X) & \to & \mathrm{CH}^r(Y) \\ \downarrow & & \downarrow \\ H^{2r}_{\mathcal{D}}(X, \mathbb{Z}(r)) & \to & H^{2r}_{\mathcal{D}}(Y, \mathbb{Z}(r)); \end{array}$$

Further, if we assume that the restriction $\xi_Y \in \mathrm{CH}_{\mathrm{hom}}(Y)$ is null-homologous, then $\mathrm{cl}_{r,0}(\xi) \in H^{2r}_{\mathcal{D}}(X, \mathbb{Z}(r)) \mapsto J^r(Y) \subset H^{2r}_{\mathcal{D}}(Y, \mathbb{Z}(r))$. Next, if $Y = X_0 \in \{X_t\}_{t\in S}$ is a family of smooth hypersurfaces of X, then such a ξ determines a holomorphically varying map $\nu_\xi(t) \in J^r(X_t)$, called a normal function. Roughly speaking, the class $\mathrm{cl}_r(\xi) = \delta(\nu_\xi) \in H^{r,r}(X, \mathbb{Z}(r))$ is called the topological invariant of ν_ξ, i.e. ν_ξ determines $\mathrm{cl}_r(\xi)$.

Example 8.5 (Griffiths' famous example ([18]))**.** Let:

$$X = V(z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5) \subset \mathbb{P}^5$$

be the Fermat quintic fourfold. Consider these 3 copies of $\mathbb{P}^2 \subset X$:

$$L_1 := V(z_0 + z_1, z_2 + z_3, z_4 + z_5),$$

$$L_2 := V(z_0 + \xi z_2, z_2 + \xi z_3, z_4 + z_5),$$

$$L_3 := V(z_0 + \xi z_1, z_2 + \xi z_3, z_4 + \xi z_5).$$

where ξ is a primitive 5-th root of unity. Then $L_3 \bullet (L_1 - L_2) = 1 \neq 0$, hence $\xi := [L_1 - L_2]$ is a non-zero class in $H^{2,2}(X, \mathbb{Z}(2))$. Further, if $\{X_t\}_{t\in U\subset\mathbb{P}^1}$ is a general pencil of smooth hyperplane sections of X, and if $t \in U$, then it is well know that $\xi_t \in \mathrm{CH}^2_{\mathrm{hom}}(X_t)$ by a theorem of Lefschetz. Since $\delta(\nu_\xi) = [L_1 - L_2] \neq 0$, it follows that $\nu_\xi(t)$ is non-zero for most $t \in U$. Therefore for very general $t \in U$, $\mathrm{Griff}^2(X_t)$ contains an infinite cycle group by Theorem 8.3. The upshot is that if:

$$Y = V\bigg(z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 + \big(\sum_{j=0}^{4} a_j z_j\big)^5\bigg) \subset \mathbb{P}^4,$$

for general $a_0, \ldots, a_4 \in \mathbb{C}$, then $\mathrm{Griff}^2(Y) \neq 0$ contains an infinite cyclic subgroup. H. Clemens was the first to show that the Griffiths group of a general quintic threefold in $\mathbb{P}^4$ is [countably] infinite dimensional, when tensored over $\mathbb{Q}$. Later it was shown by C. Voisin that the same holds for general CY threefolds. The idea is to make use of the rational curves on such threefolds.

Theorem 8.6 (See [1], [17], [19], [18], [9], [30])**.** *Let $X \subset \mathbb{P}^4$ be a (smooth) threefold of degree d. If $d \leq 4$, then $\Phi_2 : \mathrm{CH}^2_{\mathrm{hom}}(X) \xrightarrow{\sim} J^2(X)$ is an isomorphism. Now assume that X is general. If $d \geq 6$ then $Im\big(\Phi_2\big)$ is torsion. If $d = 5$, then $Im\big(\Phi_2\big) \otimes \mathbb{Q}$ is countably infinite dimensional.*

Theorem 8.7 ([30])**.** *If X is a very general Calabi-Yau threefold, then $Im\big(\Phi_2\big)$ is a countably infinite dimensional, when tensored over $\mathbb{Q}$. In particular, since $\Phi_2(\mathrm{CH}^2_{\mathrm{alg}}(X)) = 0$, it follows that $Griff^2(X; \mathbb{Q})$ is [countably] infinite dimensional over $\mathbb{Q}$.*

8.8. **Case $m=1$ and $K3$ surfaces.** Recall the Tame symbol map

$$T : \bigoplus_{\operatorname{codim}_X Z = r-2} K_2^M(\mathbb{C}(Z)) \to \bigoplus_{\operatorname{codim}_X D = r-1} K_1^M(\mathbb{C}(D)).$$

Then:

$$\mathrm{CH}^r(X,1) = H^{r-1}_{\mathrm{Zar}}(X, \mathcal{K}^M_{r,X}) \simeq \left\{ \frac{\sum_j (f_j, D_j) \bigm| \sum_j \operatorname{div}(f_j) = 0}{T\bigl(\Gamma(\bigoplus_{\operatorname{codim}_X Z = r-2} K_2^M(\mathbb{C}(Z)))\bigr)} \right\}.$$

We recall:

Definition 8.9. The subgroup of $\mathrm{CH}^r(X,1)$ represented by $\mathbb{C}^\times \otimes \mathrm{CH}^{r-1}(X)$ is called the subgroup of decomposables $\mathrm{CH}^r_{\mathrm{dec}}(X,1) \subset \mathrm{CH}^r(X,1)$. The space of indeconposables is given by

$$\mathrm{CH}^r_{\mathrm{ind}}(X,1) := \frac{\mathrm{CH}^r(X,1)}{\mathrm{CH}^r_{\mathrm{dec}}(X,1)}.$$

The map

$$\mathrm{cl}_{r,1} : \mathrm{CH}^r_{\mathrm{hom}}(X,1) \to H^{2r-1}_{\mathcal{D}}(X, \mathbb{Z}(r)),$$

is given by a map

$$\mathrm{cl}_{r,1} : \mathrm{CH}^r_{\mathrm{hom}}(X,1) \to \frac{F^{d-r+1}H^{2d-2r+2}(X,\mathbb{C})^\vee}{H_{2d-2r+2}(X, \mathbb{Z}(d-r))},$$

defined as follows. Assume given a higher Chow cycle $\xi = \sum_{i=1}^N (f_i, Z_i)$ representing a class in $\mathrm{CH}^r_{\mathrm{hom}}(X,1)$. Then via a proper modification, we can view $f_i : Z_i \to \mathbb{P}^1$ as a morphism, and consider the $2d-2r+1$-chain $\gamma_i = f_i^{-1}([-\infty, 0])$. Then $\sum_{i=1}^N \operatorname{div} f_i = 0$ implies that $\gamma := \sum_{i=1}^N \gamma_i$ defines a $2d-2r+1$-cycle. Since ξ is null-homologous, it is easy to show that γ bounds some real dimensional $2d-2r+2$-chain ζ in X, viz.,$\partial\zeta = \gamma$. For $\omega \in F^{d-r+1}H^{2d-2r+2}(X,\mathbb{C})$, the current defining $\mathrm{cl}_{r,1}(\xi)$ is given by:

$$\mathrm{cl}_{r,1}(\xi)(\omega) \;=\; \frac{1}{(2\pi\sqrt{-1})^{d-r+1}} \left[\sum_{i=1}^N \int_{Z_i \setminus \gamma_i} \omega \log f_i - \quad 2\pi\sqrt{-1}\int_\zeta \omega\right],$$

where we choose the principal branch of the log function. One can easily check that the current defined above is d-closed. Namely, if we write $\omega = d\eta$ for some $\eta \in F^{d-r+1}E_X^{2d-2r}$, then by a Stokes' theorem argument, both integrals above contribute to "periods" which cancel.

Using the description of real Deligne cohomology given above, and the regulator formula, we arrive at the formula for the real regulator $r_{r,1} : \mathrm{CH}^r(X,1) \to H^{2r-1}_{\mathcal{D}}(X, \mathbb{R}(r)) = H^{r-1,r-1}(X, \mathbb{R}((r-1)) \simeq H^{d-r+1,d-r+1}(X, \mathbb{R}(d-r+1))^\vee$. Namely:

$$r_{r,1}(\xi)(\omega) = \frac{1}{(2\pi\sqrt{-1})^{d-r+1}} \sum_j \int_{Z_j} \omega \log|f_j|.$$

Example 8.10. Suppose that X is a surface. Then we have

$$\mathrm{cl}_{2,1} : \mathrm{CH}^2_{\mathrm{hom}}(X,1) \to \frac{\{H^{2,0}(X) \oplus H^{1,1}(X)\}^\vee}{H_2(X,\mathbb{Z})}.$$

The corresponding transcendental regulator is defined to be

$$\Phi_{2,1} : \mathrm{CH}^2_{\mathrm{hom}}(X,1) \to \frac{H^{2,0}(X)^\vee}{H_2(X,\mathbb{Z})},$$

$$\Phi_{2,1}(\xi)(\omega) = \int_\zeta \omega.$$

and real regulator

$$r_{2,1} : \mathrm{CH}^2(X,1) \to H^{1,1}(X,\mathbb{R}(1))^\vee \simeq H^{1,1}(X,\mathbb{R}(1)),$$

$$r_{2,1}(\xi)(\omega) = \frac{1}{2\pi\sqrt{-1}} \sum_j \int_{Z_j} \log|f_j|\omega.$$

There is an induced map

$$\underline{r}_{2,1} : \mathrm{CH}^2_{\mathrm{ind}}(X,1) \to H^{1,1}_{\mathrm{tr}}(X,\mathbb{R}(1)).$$

If X is a $K3$ surface, then $\mathrm{CH}^2_{\mathrm{hom}}(X,1) = \mathrm{CH}^2(X,1)$, hence there is an induced map

$$\underline{\Phi}_{2,1} : \mathrm{CH}^2_{\mathrm{ind}}(X,1) \to \frac{H^{2,0}(X)^\vee}{H_2(X,\mathbb{Z})}.$$

Theorem 8.11. (i) *([26]) Let $X \subset \mathbb{P}^3$ be a smooth surface of degree d. If $d \le 3$, then $r_{2,1} : \mathrm{CH}^2(X,1) \to H^{1,1}(X,\mathbb{R}(1))$ is surjective; moreover $\mathrm{CH}^2_{\mathrm{ind}}(X,1;\mathbb{Q}) = 0$. Now assume that X is general. If $d \ge 5$, then $Im(r_{2,1})$ is "trivial".*

(ii) *[Hodge-$\mathcal{D}$-conjecture for $K3$ surfaces ([6])] Let X be a general member of a universal family of projective $K3$ surfaces, in the sense of the real analytic topology. Then*

$$r_{2,1} : \mathrm{CH}^2(X,1) \otimes \mathbb{R} \to H^{1,1}(X,\mathbb{R}(1)),$$

is surjective.

(iii) *([7]) Let $X/\mathbb{C}$ be a very general algebraic $K3$ surface. Then the transcendental regulator $\Phi_{2,1}$ is non-trivial. Quite generally, if X is a very general member of a general subvariety of dimension $20-\ell$, describing a family of $K3$ surfaces with general member of Picard rank ℓ, with $\ell < 20$, then $\Phi_{2,1}$ is non-trivial.*

Remark 8.12. (i) Regarding part (iii) of Theorem 8.11, one can ask whether $\Phi_{2,1}$ can be non-trivial for those $K3$ X with Picard rank 20, (which are rigid and therefore defined over $\overline{\mathbb{Q}}$)? In [7], some evidence is provided in support of this.

(ii) One of the key ingredients in the proof of the above theorem is the existence of plenty of nodal rational curves on a general $K3$ surface. Indeed, there is the following result:

Theorem 8.13 ([5])**.** *For a very general $K3$ surface, the set of rational on X is a dense subset in the strong (= analytic) topology.*

8.14. **Case $m = 2$ and elliptic curves.** Regulator examples on $\mathrm{CH}^k(X,2)$ Let X be a compact Riemann surface. In [25] there is constructed a real regulator

$$r : \mathrm{CH}^2(X,2) \to H^1(X,\mathbb{R}(1)), \tag{7}$$

given by

$$\omega \in H^1(X,\mathbb{R}) \simeq H^1(X,\mathbb{R}(1))^\vee \mapsto \int_X \Big[\log|f| d\log|g| - \log|g| d\log|f|\Big] \wedge \omega$$
$$= 2\int_X \log|f| d\log|g| \wedge \omega, \ \text{(by a Stokes}' \text{ theorem argument)}.$$

Alternatively, up to a twist, and real isomorphism, this is the same as the regulator $r_{2,2}$ in (4).

$$\frac{1}{2\pi\sqrt{-1}} \int_X \bigg[\log|f|\pi_1\Big(\frac{dg}{g}\Big) - \log|g|\pi_1\Big(\frac{df}{f}\Big)\bigg] \wedge \omega. \tag{8}$$

This latter formula has the following homological version (see [28]). Fix $p \in X$, and consider any loop γ in $X\backslash\{|(f)| \cup |(g)|\}$ based at p. Then via Poincaré duality $H_1(X,\mathbb{R}) \simeq H^1(X,\mathbb{R})$,

$$\gamma \mapsto \frac{1}{2\pi}\operatorname{Im}\bigg(\int_\gamma \log f \frac{dg}{g} - \log|g(p)| \int_\gamma \frac{df}{f}\bigg). \tag{9}$$

To show that the formulas (8) and (9) agree on $\mathrm{CH}^2(X,2)$, we work out the details below. First of all, observe that:

$$\pi_1\Big(\frac{dg}{g}\Big) = \pi_1(d\log g) = \sqrt{-1}d\arg(g) \quad ; \quad \operatorname{Re}\Big(\frac{dg}{g}\Big) = d\log|g|.$$

Next,

$$\operatorname{Im}\bigg[\int_\gamma (\log f)\frac{dg}{g} - \log|g(p)| \int_\gamma \frac{df}{f}\bigg]$$
$$= -\sqrt{-1}\pi_1\bigg[\int_\gamma (\log f)\frac{dg}{g} - \log|g(p)| \int_\gamma \frac{df}{f}\bigg]$$
$$= -\log|g(p)|\arg_\gamma(f) + \int_\gamma \arg(f) d\log|g| + \int_\gamma \log|f| d\arg(g).$$

Note that

$$d\big(\arg(f)\log|g|\big) = \log|g| d\arg(f) + \arg(f) d\log|g|,$$

and by Stokes' theorem:

$$\int_\gamma d\big(\arg(f)\log|g|\big) = \log|g(p)|\arg_\gamma(f).$$

Therefore:

$$\frac{1}{2\pi}\operatorname{Im}\bigg(\int_\gamma \log f \frac{dg}{g} - \log|g(p)| \int_\gamma \frac{df}{f}\bigg)$$
$$= \frac{1}{2\pi}\int_\gamma \Big[\log|f| d\arg(g) - \log|g| d\arg(f)\Big]$$
$$= \frac{1}{2\pi\sqrt{-1}}\int_\gamma \bigg[\log|f|\pi_1\Big(\frac{dg}{g}\Big) - \log|g|\pi_1\Big(\frac{df}{f}\Big)\bigg].$$

Finally, formulas (8) and (9) coincide on $\mathrm{CH}^2(X,2)$ by Poincaré duality, where $\omega := [\gamma]$, and where $[\gamma] \in H^1(X,\mathbb{R})$ is the Poincaré dual of $\gamma \in H_1(X,\mathbb{R})$.

8.15. **Bloch's construction on $K_2(X)$, for an elliptic curve X.** Let X be an elliptic curve and assume given $f,\ g \in \mathbb{C}(X)^\times$ such that $\Sigma := |\mathrm{div}(f)| \cup |\mathrm{div}(g)|$ are points of order N in $\mathrm{Pic}(X)$. Then

$$T\big(\{f,g\}^N\big) \in \coprod \mathbb{C}^\times \quad \text{and} \ \mapsto 0 \in \mathrm{Pic}(X) \otimes \mathbb{C}^\times.$$

Thus there exists $\{h_i\} \in \mathbb{C}(X)^\times$ and $\{c_i\} \in \mathbb{C}^\times$ such that $\{f,g\}^N \prod \{h_i, c_i\} \in \mathrm{CH}^2(X,2)$.[1] Note that the terms $\{h_i, c_i\}$ do not contribute to the regulator value by the formula in (6.11) above. Clearly this construction takes advantage of the existence of a dense subset of torsion points on X.

Bloch ([4]) shows that the real regulator is nontrivial for general elliptic curves, and indeed A. Collino ([11]) shows that the regulator image of $\mathrm{CH}^2(X,2)$ for a general elliptic curve X is infinite dimensional (over $\mathbf{Q}$). For curves X of genus $g > 1$, the problem of constructing classes in $\mathrm{CH}^2(X,2)$ seems to be related to the fact that under the Abel-Jacobi mapping $\Phi : X \to J^1(X), \quad p \mapsto \{p - p_0\}$, the inverse image of the torsion subgroup, $\Phi^{-1}(J^1(X)_{\mathrm{tor}})$, is finite. This is due to Raynaud's *mixed characteristic* proof of the Mumford-Manin conjecture ([29]). From a different perspective, and using infinitesimal methods, A. Collino (op. cit.) proves that for a general curve X of genus $g > 1$, the image of the regulator map $\mathrm{CH}^2(X,2) \to H^2_{\mathcal{D}}(X,\mathbb{Z}(2))$ is torsion, hence the real regulator image is zero.

On the flip side, one can arrive at an *analytic* proof of the following (weaker) version of the Mumford-Manin conjecture ([25]):

6.15 *Let $X \subset \mathbb{P}^2$ be a general curve of degree $d \geq 3$, and consider the Abel-Jacobi map $X \hookrightarrow J^1(X)$. Put $\Gamma = X \cap J^1_{\mathrm{tor}}(X)$. Then Γ is dense in $X \Leftrightarrow d = 3$.*

The basic idea of proof is this. Let $[z_0, z_1, z_2]$ be homogeneous cordinates for $\mathbf{P}^2$, and let $(x,y) = (z_1/z_0, z_2/z_0)$ be corresponding affine cordinates. Via a degeneration argument, one can show that:

$$\int_X \Big[\log|x| d\log|y| - \log|y| d\log|x|\Big] \wedge \omega \neq 0,$$

for a suitable choice of real 1-form ω on X, and for general such $X \subset \mathbb{P}^2$. It is notationally more convenient to write this as:

$$\int_X \Big[\log|f| d\log|g| - \log|g| d\log|f|\Big] \wedge \omega \neq 0,$$

[1] This uses the Weil reciprocity theorem. Let X be a compact Riemann surface, $f,g \in \mathbb{C}(X)^\times$, and for $p \in X$, write

$$T_p\{f,g\} = (-1)^{\nu_p(g)\nu_p(f)} \left(\frac{f^{\nu_p(g)}}{g^{\nu_p(f)}}\right)\Big|_p \in \mathbb{C}^\times.$$

Note that for $p \notin |\mathrm{div}(f)| \bigcup |\mathrm{div}(g)|$, we have $T_p\{f,g\} = 1$. Thus we can write $T\{f,g\} = \sum_{p\in X} T_p\{f,g\}$. Weil reciprocity says that $\prod_{p \in X} T_p\{f,g\} = 1$. Let us rewrite this as follows. If we write $T\{f,g\} = \sum_{j=1}^M (c_j, p_j)$, where $p_j \in X$ and $c_j \in \mathbf{C}^\times$, then $\prod_{j=1}^M c_j = 1$. Now fix $p \in X$ and let us suppose that $Np_j \sim_{\mathrm{rat}} Np$ for all j. Thus there exists $h_j \in \mathbb{C}(X)^\times$ such that $(h_j) = Np_j - Np$. Then $T\{h_j, c_j\} = (c_j^N, p) + (c_j^{-N}, p_j)$. The result is that $T\big(\{f,g\}^N\{h_1,c_1\}\cdots\{h_M,c_M\}\big) = \prod_{j=1}^M (c_j^N, p) = (1,p) =$ "0".

for some $f,\ g \in \mathbb{C}(X)^{\times}$. Now assume that Γ is dense in X, and choose 0-cycles $\xi_f,\ \xi_g$ "close to" $\mathrm{div}(f)$ and $\mathrm{div}(g)$ respectively, such that $\xi_f,\ \xi_g$ are supported on Γ. Thus for some integer $N > 0$, we can write $N\xi_f = (\tilde{f})$, $N\xi_g = (\tilde{g})$ for some $\tilde{f},\ \tilde{g} \in \mathbb{C}(X)^{\times}$, and for which

$$\int_X \Big[\log|\tilde{f}|d\log|\tilde{g}| - \log|\tilde{g}|d\log|\tilde{f}|\Big] \wedge \omega \neq 0, \ \text{(see [25] for details)}.$$

This leads us to the setting where we may assume that the divisor sets of f and g are torsion points, of order N say. Thus we can extend $\{f,g\}$ to a class $\underline{\xi} \in \mathrm{CH}^2(X,2)$, for which $r(\underline{\xi}) \neq 0$. However, as we mentioned above, from the work of A. Collino, the real regulator image is zero for a general curve $X \subset \mathbb{P}^2$ of degree ≥ 4, and more generally for a general curve of genus $g > 1$. Thus in our case, we have established that Γ dense in $X \Rightarrow d = 3$. The converse statement is obvious.

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