Interest Rates & Credit Risk — Solutions #2

1.1 W.L.O.G set $K=1$.

Then $E_0 = \text{BSCall}(A_0, 1, \sigma, r, T) := F(A_0, T)$

and $D_0 = A_0 - F(A_0) = LF(A_0)$ ($L = L_0 =$ leverage).

Root-finder to solve $g(A_0) := A_0 + (L-1)F(A_0) = 0$

note $g'(A_0) = 1 + (L-1)\Delta$ BS delta.

Newton-Raphson Iteration $A_0^{(i+1)} = A_0^{(i)} - \dfrac{g(A_0^{(i)})}{g'(A_0^{(i)})}$

1.2. $YS(T) = \frac{1}{T} \log \left(P_0(T) / D_0(T) \right)$

[6]

MATLAB [6]

2. $s < t < T$: $p_\tau(t \mid \mathcal{F}_s) = -\frac{\partial}{\partial t} E[1_{\{\tau > t\}} \mid \mathcal{F}_s]$

$$E[1_{\{\tau > t\}} \mid \mathcal{F}_s] = H_s^c P\left[\min_{s \le u \le t} (W_t - W_s) + m(t-s) \ge \sigma^{-1} \log\left(\frac{K(s)}{A_s}\right) \right]$$

$$= H_s^c \cdot FP(-d; -m, t-s) \qquad m = \frac{1}{\sigma}(r - \sigma^2/2 - k)$$

$$d = \frac{1}{\sigma} \log(K(s)/A_s).$$

$\therefore$ If $H_s^c = 1$ (no default before s)

$$p_\tau(t \mid \mathcal{F}_s) = \left(\frac{d}{2(t-s)} + \frac{m}{2} \right) \cdot \frac{1}{\sqrt{t-s}} \phi\left(-\frac{d + m(t-s)}{\sqrt{t-s}} \right)$$

$$- e^{-2md} \left(\frac{-d}{2(t-s)} + \frac{m}{2} \right) \cdot \frac{1}{\sqrt{t-s}} \phi\left(\frac{d + m(t-s)}{\sqrt{t-s}} \right).$$

A_s $t \downarrow s$ $p_\tau(t \mid \mathcal{F}_s) \longrightarrow 0$ as long as $A_s > K(s)$.

[8] $\therefore$ No intensity (its zero, since τ is predictable).

3. Let $X_t = \pi_t^D D_t + \pi_t^E E_t$ be wealth of a self-financing portfolio. $\begin{cases} \pi_t^D = \# \text{ of bonds at } t \\ \pi_t^E = \# \text{ of shares} \end{cases}$

Then
$$dX_t = \pi_t^D \, dD_t + \pi_t^E \, dE_t.$$
$$= \pi_t^D \left[\left(\partial_t D + \mu A_t \partial_A D + \tfrac{1}{2}\sigma^2 A_t^2 \partial_A^2 D\right) dt + \sigma A_t \partial_A D \, dW_t \right]$$
$$+ \pi_t^E \left[\left(\partial_t E + \mu A_t \partial_A E + \tfrac{1}{2}\sigma^2 A_t^2 \partial_A^2 E\right) dt + \sigma A_t \partial_A E \, dW_t \right]$$

locally risk free $\Rightarrow$ $\begin{cases} \pi_t^D \partial_A D + \pi_t^E \partial_A E = 0 & \text{(no } dW \text{ term)} \\ dX_t = r X_t \, dt \end{cases}$

Finally $D = A - E$ so $\partial_t D = -\partial_t E$, $\partial_A D = 1 - \partial_A E$, $\partial_A^2 D = -\partial_A^2 E$

$$\therefore -\pi^E \left(\frac{\partial_A E}{\partial_A D}\right)\left(\partial_t D + \mu A_t \partial_A D + \tfrac{1}{2}\sigma^2 A_t^2 \partial_A^2 D - rD\right)$$
$$+ \pi^E \left(\partial_t E + \mu A_t \partial_A E + \tfrac{1}{2}\sigma^2 A_t^2 \partial_A^2 E - rE\right) = 0$$

Substitute, multiply by $\partial_A D / \pi^E$:
$$-\partial_A E \left[-\cancel{\partial_t E} + \mu A_t \left[1 - \cancel{\partial_A E}\right] + \tfrac{1}{2}\sigma^2 \cancel{A_t^2 \partial_A^2 E} - r(A - \cancel{E}) \right]$$
$$+ (1 - \cancel{\partial_A E})\left(\partial_t E + \mu A_t \partial_A E + \tfrac{1}{2}\sigma^2 A_t^2 \partial_A^2 E - rE\right] = 0$$

Cancelling terms gives
$$0 = -\partial_A E \left[\cancel{\mu A_t} - rA \right] + \partial_t E + \cancel{\mu A_t \partial_A E} + \tfrac{1}{2}\sigma^2 A_t^2 \partial_A^2 E - rE$$
$$= \partial_t E + rA \partial_A E + \tfrac{1}{2}\sigma^2 A_t^2 \partial_A^2 E - rE.$$

Since E, A satisfy this B-S eqn, so does $D = A - E$.

[10]

4. $\underline{\text{W.L.O.G take } N=1}$

$$E_t = S_t = \mathrm{BSCall}(A_t, K, r, \sigma, T-t)$$

$$dS_t = \text{drift} + \left(\frac{\partial_A \mathrm{BScall}}{S_t}\right) \sigma A_t \, dW_t$$

$$\therefore \sigma_t^{(s)} = \frac{\partial_A \mathrm{BSCall} \cdot \sigma A_t}{S_t} = \frac{\Delta(A_t, K, r, \sigma, T-t) \cdot \sigma A_t}{S_t}$$

Let $G(S_t, T-t)$ be inverse of $\mathrm{BSCall}(A_t, \dots T-t)$
(Note BSCall is monotonic)

Then $\sigma_t^{(s)} = f(T-t, S_t)$ where

$$f(T, S) = \frac{\Delta(G(S,T)) \cdot \sigma G(S,T)}{S}$$

$\underline{\text{Note}}$ $\Delta = N(d_1)$ (as usual)

Since $e^{-rt} S_t$ is Q-mg

$$dS_t = r S_t \, dt + \sigma_t^{(s)} S_t \, dW_t$$

Since this is Itô diffusion, S is strongly Markov

$\underline{\text{leverage} \uparrow \infty \text{ as } A_0 \downarrow 0}$ (K fixed).

[6] One can show $f(T, S) \sim \dfrac{d_2}{d_2 - d_1} \longrightarrow +\infty$ (but this is hard!)

5. • $P[\tau^{(1)} > t] = P[N_t = 0] = e^{-\lambda t}$

$\therefore \tau^{(1)} \sim \mathrm{Exp}(\lambda)$ $\quad \mathrm{PDF} = -\frac{\partial}{\partial t}(e^{-\lambda t}) = \lambda e^{-\lambda t}$

[6] • $P[\tau^{(n)} > t] = P[N_t < n] = \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}$

$\underline{\text{MATLAB}}$ [4] $\quad \mathrm{PDF} = -\frac{\partial}{\partial t}(\quad) = \sum_{k=0}^{n-1} \left(\lambda - \frac{k}{t}\right) \frac{(\lambda t)^k e^{-\lambda t}}{k!}$

$$= \lambda \left[\sum_{k=0}^{n-1} \frac{(\lambda t)^k e^{-\lambda t}}{k!} - \sum_{k=0}^{n-2} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \right] = \frac{\lambda (\lambda t)^{n-1} e^{-\lambda t}}{(n-1)!}$$

6. Use (5.21) with $Y = e^{-\int_s^t r_u\,du}$ ($\mathcal{G}$-measurable)

Then $H_s^c \overline{P_s}(t) := E^{Q}\left[H_t^c \, e^{-\int_s^t r_u\,du} \,\middle|\, \mathcal{F}_s\right]$ $(s < t)$

[4] $= H_s^c \, E\left[e^{-\int_s^t r_u\,du} \cdot e^{-\int_s^t \lambda_u\,du} \,\middle|\, \mathcal{G}_s\right]$. <u>as needed!</u>