On the C^*-algebras of homoclinic and heteroclinic equivalence in expansive dynamics

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Based on

C^*-algebras of homoclinic and heteroclinic structure in expansive dynamics

which you can download from http://www.imf.au.dk/en/

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C^*-algebras of homoclinic and heteroclinic structure in expansive dynamics

which you can download from http://www.imf.au.dk/en/ and

The homoclinic and heteroclinic C^*-algebras of a generalized one-dimensional solenoid

which is in preparation.

Expansiveness

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ψ is *expansive* when there is a $\delta > 0$ such that

$$x \neq y \ \Rightarrow d\left(\psi^k(x), \psi^k(y)\right) \geq \delta \text{ for some } k \in \mathbb{Z}.$$

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x, y are *strongly homoclinic* when there are open neighborhoods U, V of x and y, respectively, and a homeomorphism $\chi : U \to V$, called a local conjuagcy, such that $\chi(x) = y$ and

$$\lim_{k \to \pm\infty} \sup_{z \in U} d\left(\psi^k(z), \psi^k\left(\chi(z)\right)\right) = 0.$$

The homoclinic algebra

Unlike homoclinicity, strong homoclinicity is always an étale equivalence relation

$$R = \left\{(x,y) \in X^2 : \ x \text{ and } y \text{ are strongly homoclinic } \right\}.$$

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The homoclinic algebra is $A_\psi(X) = C^*_r(R)$ (Renault)

$$fg(x,y) = \sum_z f(x,z)g(z,y)$$

$$f^*(x,y) = \overline{f(y,x)}.$$

D. Ruelle, I. Putnam (for Smale spaces): The asymptotic algebra.

Post-periodic points and the Wagoner topology

$x \in X$ is *post-periodic* when there is a ψ-periodic point $p \in X$ such that $\lim_{k \to -\infty} d\left(\psi^k(x), \psi^k(p)\right) = 0$.

W - the set of post-periodic points.

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W is a locally compact Hausdorff space in a topology - *the Wagoner topology*- where a neigborhood base of a point $x \in W$ is given by the sets

$$\left\{y \in X : \ d\left(\psi^k(x), \psi^y(y)\right) < \epsilon, \ j \leq k\right\}, \quad k \in \mathbb{Z}, \ \epsilon \in \left]0, \epsilon_x\right[.$$

The heteroclinic algebra

$x, y \in W$ are *locally conjugate* or *strongly heteroclinic* when there are open neighborhoods U, V of x and y in W, respectively, and a homeomorphism $\chi : U \to V$ such that $\chi(x) = y$ and

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The reduced groupoid C^*-algebra of this equivalence relation is the *the heteroclinic algebra* $B_\psi(X)$.

For Smale spaces $B_\psi(X)$ is stably isomorphic to the stable algebra of I. Putnam.

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α) (Expansion) There are constants $C > 0$ and $\lambda > 1$ such that

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for every $n \in \mathbb{N}$ when $x, y \in e \in \mathbb{E}$ and there is an edge $e' \in \mathbb{E}$ with $h^n\left([x, y]\right) \subseteq e'$.

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- γ) (Markov) $h(\mathbb{V}) \subseteq \mathbb{V}$.
- δ) (Mixing) For every edge $e \in \mathbb{E}$ there is an $m \in \mathbb{N}$ such that $\Gamma \subseteq h^m(e)$.
- ϵ) (Flattening) There is a $d \in \mathbb{N}$ such that for all $x \in \Gamma$ there is a neighborhood U_x of x with $h^d(U_x)$ homeomorphic to $]-1, 1[$.

Generalized one-solenoids

Set

$$\overline{\Gamma} = \left\{ (x_i)_{i=0}^{\infty} \in \Gamma^{\mathbb{N}} : \ h(x_{i+1}) = x_i, \ i = 0, 1, 2, \dots \right\}.$$

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Define $\overline{h} : \overline{\Gamma} \to \overline{\Gamma}$ such that $\overline{h}(x)_i = h(x_i)$ for all $i \in \mathbb{N}$.

$\overline{h}$ is a homeomorphism with inverse

$$\overline{h}^{-1}(z_0, z_1, z_2, \dots) = (z_1, z_2, z_3, \dots) .$$

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Most of the following theorem is due to I.Yi.

Theorem

A generalized one-solenoid $\left(\overline{\Gamma}, \overline{h}\right)$ *is an expansive homeomorphism. It is mixing and a Smale space.*

The heteroclinic algebra of the inverse of a generalized one-solenoid

This is the 'unstable algebra' in the terminology of I. Putnam.

Theorem

The heteroclinic algebra $B_{\overline{h}^{-1}}\left(\overline{\Gamma}\right)$ *is isomorphic to*

$$\mathbb{K} \otimes \left(C(K) \rtimes_{\psi} \mathbb{Z}\right),$$

where ψ *is a minimal homeomorphism of the Cantor set* K*. (*ψ *is either an odometer or a primitive substitution.)*

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In particular, it follows that $B_{\overline{h}^{-1}}\left(\overline{\Gamma}\right)$ is a simple AT-algebra of real rank zero with an essentially unique lower semi-continuous densely defined trace. $K_1\left(B_{\overline{h}^{-1}}\left(\overline{\Gamma}\right)\right) = \mathbb{Z}$ and $K_0\left(B_{\overline{h}^{-1}}\left(\overline{\Gamma}\right)\right)$ is a stationary dimension group.

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Names: I. Yi, R. Herman, I. Putnam, C. Skau, T. Giordano (and others).

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$$K_1\left(B_{\overline{h}}\left(\overline{\Gamma}\right)\right) = \mathbb{Z} \qquad \text{or} \qquad K_1\left(B_{\overline{h}}\left(\overline{\Gamma}\right)\right) = \mathbb{Z}_2.$$

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There are examples where both $K_1\left(A_{\overline{h}}\left(\overline{\Gamma}\right)\right)$ and $K_0\left(A_{\overline{h}}\left(\overline{\Gamma}\right)\right)$ contains two-torsion.

By relating to the work of I. Putnam we find that

Theorem

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The homoclinic algebra $A_{\overline{h}}\left(\overline{\Gamma}\right)$ *is stably isomorphic to* $B_{\overline{h}}\left(\overline{\Gamma}\right) \otimes B_{\overline{h}^{-1}}\left(\overline{\Gamma}\right)$.

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Strategy: Study $B_{\overline{h}}\left(\overline{\Gamma}\right)$ and $B_{\overline{h}^{-1}}\left(\overline{\Gamma}\right)$ seperately.

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Focus on $B_{\overline{h}}(\overline{\Gamma})$.

First observation: Two points $(x_0, x_1, x_2, \dots), (y_0, y_1, y_2, \dots) \in \overline{\Gamma}$ are forward asymptotic under $\overline{h}$ if and only if

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for some $k \in \mathbb{N}$.

On the proofs

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Second observation: If a, b are elements in $\Gamma \backslash \mathbb{V}$ and $h^k(a) = h^k(b)$, there is an $m > k$ and open neighborhoods U_a and U_b of a and b, respectively, such that $h^m(U_a) = h^m(U_b) \simeq]-1, 1[$.

On the proofs - open interval graph relations

Let $g : [-1, 1] \to \Gamma$ be a continuous locally injective map. Define an equivalence relation $\sim$ on $]-1, 1[$ such that $s \sim t$ if and only if there are open neighborhoods U_s and U_t of s and t in $]-1, 1[$ such that $g(U_s) = g(U_t) \simeq]-1, 1[$.

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$B_{\overline{h}}(\overline{\Gamma})$ is the inductive limit of a sequence

$$C_r^*(R_1) \subseteq C_r^*(R_2) \subseteq C_r^*(R_3) \subseteq \dots$$

where each R_i is an open interval graph relation.

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Thus $B_{\overline{h}}\left(\overline{\Gamma}\right)$ is a simple ASH-algebra - but we don't know much about those in general, do we?

On the proofs - a stationary system

There are $n, m \in \mathbb{N}$ and two $n \times m$ matrices I, U such that

$$I_{ij}, U_{ij} \in \{0, 1\}$$

and

$$\sum_{i=1}^{n} I_{ik} + \sum_{i=1}^{n} U_{ik} = 2.$$

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When $a = (a_1, \ldots, a_m) \in \mathbb{N}^m, b = (b_1, b_2, \ldots, b_n) \in \mathbb{N}^n$ set

$$F_a = M_{a_1} \oplus M_{a_2} \oplus \cdots \oplus M_{a_m}, \quad F_b = M_{b_1} \oplus M_{b_2} \oplus \cdots \oplus M_{b_n}.$$

Assuming that $\sum_{k=1}^{m} U_{ik} a_k = \sum_{k=1}^{m} I_{ik} a_k = b_i$, there are unital homomorphisms $\varphi_I, \varphi_U : F_a \to F_b$ with multiplicity matrices U and I.

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Set $A(a, b, U, I) = \{(x, f) \in F_a \oplus C([0,1], F_b) : \ f(0) = \varphi_I(x), \ f(1) = \varphi_U(x)\}$.

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$B_{\overline{h}}\left(\overline{\Gamma}\right)$ is the stabilized algebra of the inductive limit of a unital sequence

$$A\left(a_1, b_1, U, I\right) \xrightarrow{\pi_1} A\left(a_2, b_2, U, I\right) \xrightarrow{\pi_2} A\left(a_3, b_3, U, I\right) \xrightarrow{\pi_3} \dots$$

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A thorough study of this 'stationary' system leads to the crucial

Lemma

$B_{\overline{h}}\left(\overline{\Gamma}\right)$ has real rank zero.

On the proofs - connection to Lins work

It follows then from work of H. Lin that $B_{\overline{h}}\left(\overline{\Gamma}\right)$ has tracial rank zero, and is classified by K-theory - provided only that there are not too many traces.

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Another look a the 'stationary sequence' shows that there is in fact only one. - The rest is easy.

The contact with results from the classification community is made via Lin's results on 'tracial rank zero'. Furthermore, the proof of RR $= 0$ uses Lin's theorem on almost normal matrices. But work of Elliott, Gong, Li, Phillips is also used.