GRADUATE COURSE ON FREE PROBABILITY
PROBLEM SET 2
DUE ON OCTOBER 18, 2007

(1) Let u be a generator of a free group $\mathbb{F}_n$, realized as left multiplication operator in the canonical representation of $\mathbb{F}_n$ on $l_2(\mathbb{F}_n)$. By τ we denote the vector state corresponding to the neutral element $e \in \mathbb{F}_n$, $\tau(a) = \langle \delta_e, a\delta_e \rangle$.
(i) Calculate the moments of $u + u^*$ with respect to τ.
(ii) Determine a measure μ on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} t^n d\mu(t) = \tau[(u + u^*)^n] \qquad \text{for all } n \in \mathbb{N}$$

Is this μ unique?

(2) (i) Let $\{a_1, a_2\}$, $\{b_1, b_2\}$, $\{c_1, c_2\}$ be free (with respect to a state φ). Calculate $\varphi(a_1 b_1 c_1 c_2 a_2 b_2)$.
(ii) Prove that functions of freely independent random variables are freely independent: if a and b are freely independent and f and g polynomials, then $f(a)$ and $g(b)$ are freely independent, too.

(3) (i) Let $(\mathcal{A}, \varphi)$ be a C^*-probability space, and let $(\mathcal{A}_i)_{i\in I}$ be a freely independent family of unital $*$-subalgebras of $\mathcal{A}$. For every $i \in I$, let $\mathcal{B}_i$ be the closure of $\mathcal{A}_i$ in the norm topology. Prove that the algebras $(\mathcal{B}_i)_{i\in I}$ are freely independent.
(ii) Formulate and prove the corresponding statement for von Neumann algebras.

(4) Let b be a symmetric Bernoulli variable, i.e. a selfadjoint random variable whose distribution is the probability measure $\frac{1}{2}(\delta_{-1} + \delta_1)$. In terms of moments this means:

$$\varphi(b^n) = \begin{cases} 1, & \text{if } n \text{ even} \\ 0, & \text{if } n \text{ odd} \end{cases}$$

Show that the free cumulants of b are given by the following formula:

$$\kappa_n(b, \dots, b) = \begin{cases} (-1)^{k-1} C_{k-1}, & \text{if } n = 2k \text{ even} \\ 0, & \text{if } n \text{ odd,} \end{cases}$$

where C_k is the k-th Catalan number.

(5) In this exercise we want to establish the connection between the combinatorial description of classical cumulants and the more common formulation in terms of Fourier transforms, as already addressed in Lecture 1. We restrict here to the case of one random variable.

(i) Let $(\alpha_n)_{n\geq 1}$ and $(k_n)_{n\geq 1}$ be two sequences of numbers. Consider the exponential generating power series

$$A(z) := 1 + \sum_{n=1}^{\infty} \frac{\alpha_n}{n!} z^n, \qquad \text{and} \qquad B(z) := \sum_{n=1}^{\infty} \frac{k_n}{n!} z^n.$$

Show that the combinatorial relation

$$\alpha_n = \sum_{\pi \in \mathcal{P}(n)} k_\pi$$

between the coefficients of these power series is equivalent to the relation

$$B(z) = \log(A(z))$$

between the power series themselves.
[Hint: One possibility is to find the formal power series for $\exp f(t)$ when $f(t)$ is a power series without constant term 1, and to identify the coefficients appearing there with the number of partitions with given block sizes ... Another possibility is to follow the strategy for the proof of the corresponding statement for free cumulants: from the combinatorial relation $\alpha_n = \sum_{\pi \in \mathcal{P}(n)} k_\pi$ derive first a recursive relation between the α's and the k's and then translate this into a relation (this time a differential equation) between the two power series.]

(ii) Use the previous part of this exercise to prove the following. Let ν be a compactly supported probability measure on $\mathbb{R}$ and $\mathcal{F}$ its Fourier transform, defined by

$$\mathcal{F}(t) := \int_{\mathbb{R}} e^{itx} d\nu(x).$$

Let $(\alpha_n)_{n\geq 1}$ be the moments of ν and define its classical cumulants $(k_n)_{n\geq 1}$ by the relation $\alpha_n = \sum_{\pi \in \mathcal{P}(n)} k_\pi$. Prove that we have the power series expansions

$$\mathcal{F}(t) = 1 + \sum_{n=1}^{\infty} \frac{(it)^n}{n!} \alpha_n \qquad \text{and} \qquad \log \mathcal{F}(t) = \sum_{n=1}^{\infty} \frac{(it)^n}{n!} k_n.$$