Semilinear pseudo-differential equations and travelling waves *

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Abstract

We report on some recent results obtained by Cappiello, Gramchev and Rodino, concerning semilinear pseudo-differential equations. The linear parts of such equations are given by the so-called SG-pseudodifferential operators, introduced by Parenti and Cordes, and then studied further by Coriasco, Schulze and many others. These operators are defined on the whole Euclidean space, and a suitable ellipticity condition at infinity (SG-ellipticity) implies for them the Fredholm property in a scale of weighted Sobolev spaces. In particular one obtains that bounded eigenfunctions belong to the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. Here we prove a more precise result on exponential decay and holomorphic extension, namely eigenfunctions of linear SG-elliptic equations belong to the Gelfand-Shilov space of order (1,1). The result extends to semilinear perturbations by a technique of a priori estimates. Applications concern solitary travelling waves. In particular, the celebrated KdV equation reduces, for travelling waves, to the Newton equation, basic example of semilinear SG ordinary differential equation. Similar examples are provided by higher order travelling wave equations and stationary solutions of semilinear Schrödinger equations in higher dimension. Also some non-local models in Fluid Dynamics, suggested by Bona, provide travelling waves equations, which can be seen as semilinear perturbations of SG-elliptic pseudo-differential operators.

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1 Travelling solitary waves

The first documentation of the existence of shallow water waves appeared in 1834 when J. Scott Russell wrote one of the most cited papers about what later became known as soliton theory. Russell observed propagation of a solitary wave in the Glasgow-Edinburgh canal. In 1895 Korteweg and De Vries derived an equation describing shallow water waves, and gave the following interpretation of the solitary wave of Scott Russell. Ignoring some relevant physical aspects and simplifying parameters, we may write for short the KdV equation as

$$v_t + 2vv_x + v_{xxx} = 0, \tag{1.1}$$

where t is the time variable, x the point in the canal, $v(x,t)$ the height of the water (let us address to [3], [17], [18] for a much more detailed presentation). Looking for a solitary wave solution, travelling forward with velocity $V > 0$, we impose $v(t,x) = u(x - Vt)$ in (1.1) and we obtain

$$\frac{d}{dx}(-Vu + u^2 + u'') = 0$$

hence $u(x)$ satisfies $u'' - Vu + u^2 = \text{const}$. Assuming further const $= 0$, we are reduced to solve

$$u'' - Vu + u^2 = 0, \tag{1.2}$$

sometimes called Newton equation. Equation (1.2) possesses explicit solutions in terms of special functions. If we impose $u(x) \to 0$ for $x \to \pm\infty$, we obtain simply translations of the function

$$u(x) = \frac{\frac{3}{2}V}{\mathrm{Ch}^2\left(\frac{\sqrt{V}}{2}x\right)}, \tag{1.3}$$

where

$$\mathrm{Ch}\,t = \frac{e^t + e^{-t}}{2}.$$

We emphasize two properties of $u(x)$ in (1.3): first, it can be extended as analytic function in a strip of the form $\{z \in \mathbb{C} : |\Im z| < a\}$ in the complex plane. Second property is the exponential decay for $x \to \pm\infty$. After KdV equation, several related models were proposed. In particular recently, the theory of the solitary waves had impressive developments, both concerning applicative aspects and mathematical analysis. Let us mention applications to internal water waves, nerve pulse dynamics, ion-acoustic waves in plasma, population dynamics, etc. From the mathematical point of view, holomorphic extension and exponential decay in general situations were studied by Bona and Li [3], Biagioni and Gramchev [4], Gramchev [12] and others. In this order of ideas, we observe in particular that during the years 1990-2000, several papers were devoted to 5th order and 7th order generalization of KdV, see [17], Chapter 1. The corresponding equation (1.2) is of the type

$$\sum_{j=0}^{N} a_j u^{(j)} + Q(u) = 0 \tag{1.4}$$

where Q is a polynomial, $Q(u) = \sum_{j=2}^{M} b_j u^j$ and $a_0 = -V \neq 0$. Because of physical assumptions, the equation $\sum_{j=0}^{N} a_j \lambda^j = 0$ has no purely imaginary roots, and then all the solutions of the corresponding linear equation have exponential decay/growth. Non-trivial solutions u of (1.4) with $u(x) \to 0$ for $x \to \pm\infty$ may exist or not, according to the coefficients a_j, b_j, and when they exist, in general they do not have an explicit analytic expression. Exponential decay and holomorphic extensions are granted anyhow, by the previous theoretical results, see the next pages. Let us emphasize that, to reach the exponential decay, the boundedness of $u(x)$ is not sufficient as initial assumption. We shall express later a precise threshold in terms of Sobolev estimates; as counter-example, consider here the celebrated Burger's equation (1948):

$$v_t + v_{xx} + 2vv_x = 0. \tag{1.5}$$

Imposing $v(t,x) = u(x - Vt)$ and arguing as before we obtain the Verhulst equation

$$u' - Vu + u^2 = 0 \tag{1.6}$$

which can be regarded as particular case of (1.4). It admits the bounded solution

$$u(x) = \frac{V}{1 + e^{-Vx}}. \tag{1.7}$$

Assuming $V > 0$, we have exponential decay only for $x \to -\infty$, whereas $u(x) \to V \neq 0$ as $x \to +\infty$.

2 Semilinear Schrödinger equations

As n-dimensional generalization of (1.2), setting for simplicity $V = 1$, we may consider

$$-\Delta u + u = u^p \tag{2.1}$$

for an integer $p \geq 2$. Such equations in $\mathbb{R}^n$ have been largely studied. From the point of view of the Mathematical Physics, they appear for example when considering nonlinear Schrödinger equations used in Plasma Physics and Nonlinear Optics. Travelling waves, in this case, have to be understood as stationary wave solutions, defined as time-modulation of $v(x)$. From the point of view of the Mathematical Analysis, we address to the recent book [1] for a collection of results of existence and uniqueness, or multiplicity, of the Sobolev solutions of (2.1) via variational methods. Concerning exponential decay, we quote the following precise result in [2]. Assume $n \geq 3$. If $1 < p < \frac{n+2}{n-2}$, then (2.1) has a (positive) radial solution $u(x) = U(|x|) \in H^1(\mathbb{R}^n) = W^{1,2}(\mathbb{R}^n)$. Such a solution is unique and

$$U(r) \sim r^{-\frac{n-1}{2}} e^{-r}, \qquad r = |x| \quad as \quad r \to +\infty.$$

We address to the next sections for exponential decay and holomorphic extension of general solutions of (2.1).

3 Gelfand-Shilov spaces

A precise functional frame, where we may read the previous properties of travelling waves, is given by the Gelfand-Shilov classes, subspaces of the Schwartz class $\mathcal{S}(\mathbb{R}^n)$. Let us recall their definition. The space $S^\mu_\nu(\mathbb{R}^n), \mu > 0, \nu > 0, \mu + \nu \geq 1$, is defined as the space of all functions $f \in C^\infty(\mathbb{R}^n)$ satisfying the following estimate

$$\sup_{x \in \mathbb{R}^n} \exp(\varepsilon |x|^{1/\nu}) \, |\partial^\alpha_x f(x)| \leq C^{|\alpha|+1} (\alpha!)^\mu \tag{3.1}$$

for all $\alpha \in \mathbb{Z}^n_+$ and for some positive constants C, ε independent of α, or equivalently

$$\sup_{\alpha, \beta \in \mathbb{Z}^n_+} C^{-|\alpha|-|\beta|} (\alpha!)^{-\mu} (\beta!)^{-\nu} \sup_{x \in \mathbb{R}^n} |x^\beta \partial^\alpha_x f(x)| < +\infty \tag{3.2}$$

for a new constant $C > 0$. These spaces, introduced by Gelfand and Shilov in the book [11] (see also Mityagin [14], Pilipovic [16]), give simultaneous information on the regularity and the decay at infinity of their elements. For $\mu < 1$ the estimates (3.1), (3.2) grant that f can be extended in $\mathbb{C}^n$ as an entire function satisfying uniform decay estimates in conic neighborhoods of the real axis, see [11] for precise statements. For $\mu = 1$, f is real analytic and admits a holomorphic extension only in a strip of the form $\{z \in \mathbb{C} : |\Im z| < T\}, T > 0$. We also recall that the Fourier transformation $\mathcal{F}$ acts as an isomorphism

$$\mathcal{F} : S^\mu_\nu(\mathbb{R}^n) \longrightarrow S^\nu_\mu(\mathbb{R}^n). \tag{3.3}$$

By using Fourier transform, the functions $f \in S^\mu_\nu(\mathbb{R}^n)$ can be characterized by imposing simultaneously

$$|f(x)| \leq C e^{-\varepsilon |x|^{1/\nu}}, \qquad |\hat{f}(\xi)| \leq C e^{-\varepsilon |\xi|^{1/\mu}} \tag{3.4}$$

for some $C > 0, \varepsilon > 0$. In the following, our attention will be fixed in the case $\mu = 1, \nu = 1$. In fact, $u(x)$ in (1.3) belongs to $S^1_1(\mathbb{R})$, and the same will be shown for the other travelling waves of the previous sections.

4 SG-elliptic partial differential equations

In order to find a general class of equations and a general result, including the models in Sections 1, 2, we first discuss the linear part of the operators. Writing $D_{x_j} = -i\partial_{x_j}$ and $D^\alpha = D^{\alpha_1}_{x_1} ... D^{\alpha_n}_{x_n}$ in $\mathbb{R}^n$, we first consider the partial differential operator with constant coefficients

$$P = \sum_{|\alpha| \leq m} c_\alpha D^\alpha \tag{4.1}$$

and the corresponding symbol

$$p(\xi) = \sum_{|\alpha| \leq m} c_\alpha \xi^\alpha.$$

We assume for some $c > 0$

$$|p(\xi)| \geq c(1 + |\xi|)^m, \qquad \xi \in \mathbb{R}^n. \tag{4.2}$$

This interpretes the hypothesis for (1.4), which we may read as $p(\lambda) = \sum_{j=0}^{N} a_j(i\lambda)^j \neq 0$ for $\lambda \in \mathbb{R}$, hence $|p(\lambda)| \geq c(1+|\lambda|)^m$. Note also that the symbol of the linear part in (2.1) is $p(\xi) = |\xi|^2 + 1$, which obviously satisfies (4.2). A further generalization to linear operators with smooth variable coefficients in $\mathbb{R}^n$ is given by the so-called SG operators (see [9], [10], [15]). Let us limit attention initially to operators with polynomial coefficients:

$$P = \sum_{\substack{|\alpha| \leq m_1 \\ |\beta| \leq m_2}} c_{\alpha\beta} x^\beta D^\alpha \tag{4.3}$$

with symbol

$$p(x, \xi) = \sum_{\substack{|\alpha| \leq m_1 \\ |\beta| \leq m_2}} c_{\alpha\beta} x^\beta \xi^\alpha. \tag{4.4}$$

The natural generalization of (4.2) is the SG-ellipticity condition

$$|p(x, \xi)| \geq c(1 + |\xi|)^{m_1}(1 + |x|)^{m_2}, \qquad |x| + |\xi| \geq R, \tag{4.5}$$

with $c > 0, R > 0$. This grants a general result of rapid decay for $x \to \infty$ of the solutions of the corresponding linear equations, see [9], [10], [15] for the proof.

Theorem 4.1. *Let $u \in \mathcal{S}'(\mathbb{R}^n)$ be a solution of $Pu = f$, with P as in* (4.3)*, satisfying* (4.5) *and $f \in \mathcal{S}(\mathbb{R}^n)$. Then $u \in \mathcal{S}(\mathbb{R}^n)$. In particular $Pu = 0$ implies $u \in \mathcal{S}(\mathbb{R}^n)$.*

Looking for exponential decay and holomorphic extensions, we have the following more precise result, due to [6], [8].

Theorem 4.2. *Let $u \in \mathcal{S}'(\mathbb{R}^n)$ be a solution of $Pu = f$, with P as in Theorem 4.1 and $f \in S_1^1(\mathbb{R}^n)$. Then $u \in S_1^1(\mathbb{R}^n)$. In particular, $Pu = 0$ implies $u \in S_1^1(\mathbb{R}^n)$.*

Note that in the linear case our initial assumption is $u \in \mathcal{S}'(\mathbb{R}^n)$, so counter-examples of type (1.6), (1.7) do not take place. Finally we pass to consider semilinear equations

$$Pu = Q(u) + f, \tag{4.6}$$

with P as in (4.3), (4.5), $f \in S_1^1(\mathbb{R}^n)$ and

$$Q(u) = \sum_{j=2}^{M} b_j u^j. \tag{4.7}$$

Theorem 4.3. *Consider the equation* (4.6)*. Assume $u \in H^s(\mathbb{R}^n), s > n/2$. In the case $m_2 = 0$, that is P as in* (4.1), (4.2)*, assume further $\langle x \rangle^\varepsilon u \in H^s(\mathbb{R}^n), s > n/2$, for some $\varepsilon > 0$, where we denote $\langle x \rangle = (1 + |x|^2)^{1/2}$. Then $u \in S_1^1(\mathbb{R}^n)$.*

The previous Theorem 4.3 applies obviously to the equations of Sections 1 and 2. The proof in the case $m_2 > 0$ was given in [6].

5 Non-local travelling waves

Non-local equations, i.e. nonlinear partial differential equations involving integral operators, have been proposed as models for different solitary waves phenomena. Let us fix here attention on the so-called intermediate long equation, see [13] and recent contributions by J.L. Bona, J. Albert and others:

$$v_t + 2vv_x - (Nv)_x + v_x = 0, \tag{5.1}$$

where N is the Fourier multiplier operator defined by

$$(Nv)\hat{}(\xi) = \xi \,\mathrm{Ctgh}\, \xi \hat{v}(\xi). \tag{5.2}$$

Looking for solutions $v(t,x) = u(x - Vt)$ and arguing as before, we obtain the non-local equation

$$Nu + \gamma u = u^2 \tag{5.3}$$

where $\gamma = V - 1$. Under the assumption $V > 0$ a solution is given by

$$u(x) = \frac{a \sin a}{\mathrm{Ch}\,(ax) + \cos a}$$

where a is determined by the equation $a\,\mathrm{ctg}\ a = \gamma$. We have $\mathrm{Ch}\,(ax) + \cos a > 0$ and $u \in S_1^1(\mathbb{R})$.

6 SG-elliptic pseudo-differential equations

Theorems 4.2, 4.3 can be extended to a class of pseudo-differential operators P, to include the model of Section 5 as a particular example. For the sake of simplicity, we limit ourselves to consider Fourier multipliers:

$$p(D)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(\xi)\hat{u}(\xi) d\xi, \tag{6.1}$$

where $p(\xi)$ is a classical analytic symbol of order $m > 0$:

$$\left| D_\xi^\alpha p(\xi) \right| \le C^{|\alpha|+1} \alpha! (1 + |\xi|)^{m-|\alpha|} \tag{6.2}$$

The assumption of SG-ellipticity is formally as in (4.2):

$$|p(\xi)| \ge c(1 + |\xi|)^m, \qquad \xi \in \mathbb{R}^n. \tag{6.3}$$

The symbol of the linear part of (5.3) is

$$p(\xi) = \xi \,\mathrm{Ctgh}\, \xi + \gamma, \quad \gamma > -1,$$

and (6.2), (6.3) are satisfied.

Theorem 6.1. *Let $p(\xi)$ be SG-elliptic and consider the equation*

$$p(D)u = Q(u) + f$$

where Q is a polynomial, cf. (4.7)*, and $f \in S_1^1(\mathbb{R}^n)$. Assume that for some $\varepsilon > 0$ we have $\langle x \rangle^\varepsilon u \in H^s(\mathbb{R}^n), s > n/2$. Then $u \in S_1^1(\mathbb{R}^n)$.*

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